Arthur packets and generalized endoscopy for real reductive groups

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Goals

- Introduce a generalization (over $\mathbb{R}$) of endoscopic lifting.
- Use this generalization to prove the unitarity of *almost all* Arthur packets.

Joint work with Jeff Adams and David Vogan.

Notation

$$\begin{aligned}
G &= \text{complex connected reductive algebraic group, equipped} \\
&\quad \text{with inner class of real forms} \\
\Pi(G) &= \text{equivalence classes of irreducible admissible reps} \\
&\quad \text{of real forms of } G \text{ in specified inner class} \\
{}^\vee G &= \text{complex Langlands dual group of } G \\
{}^\vee G^\Gamma &= \text{L-group of } G \\
W_\mathbb{R} &= \text{Weil group of } \mathbb{R} \\
\Phi({}^\vee G^\Gamma) &= \{\text{Langlands parameters } \varphi : W_\mathbb{R} \to {}^\vee G^\Gamma\}
\end{aligned}$$

Langlands classification

Theorem (Langlands, Knapp-Zuckerman, Harish-Chandra)

- For each $\varphi \in \Phi({}^\vee G^\Gamma)/{}^\vee G$, there is a finite subset
$$\Pi^L_\varphi(G) \subset \Pi(G)$$
called an 'L-packet' such that
$$\Pi(G) = \bigsqcup \Pi^L_\varphi(G).$$

- For a fixed $\varphi \in \Phi({}^\vee G^\Gamma)$, there is a bijection
$$\Pi^L_\varphi(G) \xrightarrow{\sim} \{\text{irreps of } A(\varphi) := ({}^\vee G)^\varphi/({}^\vee G)^\varphi_0\}$$

Arthur parameters

Definition (Arthur)

An *Arthur parameter* is a continuous homomorphism

$$\psi : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \to {}^{\vee}G^{\Gamma}$$

such that

(a) $\psi|_{W_{\mathbb{R}}}$ is a tempered Langlands parameter, and

(b) $\psi|_{SL(2)}$ is holomorphic.

Write

$$\Psi({}^{\vee}G^{\Gamma}) := \{\text{Arthur parameters } \psi : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \to {}^{\vee}G^{\Gamma}\}.$$

Arthur parameters

We say that ψ is

- *unipotent* if $\psi(\mathbb{C}^\times) \subset Z({}^\vee G)$;
- *distinguished* if $\psi(SL(2,\mathbb{C}))$ is not contained in a proper Levi subgroup.

Note:

- distinguished $\implies$ unipotent.
- If G is semisimple, there are finitely many unipotent (hence distinguished) parameters, up to conjugation by ${}^\vee G$.

From Arthur to Langlands

To each Arthur parameter $\psi \in \Psi({}^\vee G^\Gamma)$, we can associate a Langlands parameter

$$\varphi_\psi : W_{\mathbb{R}} \to {}^\vee G^\Gamma, \qquad \varphi_\psi(w) = \psi(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix})$$

This defines an injection

$$\Psi({}^\vee G^\Gamma)/{}^\vee G \hookrightarrow \Phi({}^\vee G^\Gamma)/{}^\vee G$$

Arthur's conjectures

Theorem (ABV)

- For each $\psi \in \Psi({}^\vee G^\Gamma)/{}^\vee G$, there is a finite subset

$$\Pi_\psi^A(G) \subset \Pi(G)$$

called an 'Arthur packet' such that $\Pi_\psi^A(G) \supseteq \Pi_{\varphi_\psi}(G)$.

- For a fixed parameter $\psi \in \Psi({}^\vee G^\Gamma)$, there is a map

$$Art : \Pi_\psi^A(G) \to \{\text{fin-dim reps of } A(\psi) := ({}^\vee G)^\psi/({}^\vee G)_0^\psi\}.$$

Note: for classical groups, these are essentially the packets defined by Arthur in 2013. In particular, they are unitary.

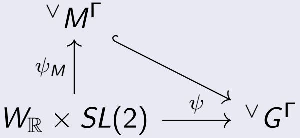
Arthur's conjectures

Conjecture (Arthur, 1983)

Each $\Pi_\psi^A(G)$ consists of unitary representations.

Theorem (Adams-MB-Vogan)

Arthur's unitarity conjecture can be reduced to the case of distinguished Arthur parameters. More precisely, suppose ψ is an Arthur parameter which factors through a *unipotent* Arthur parameter for a Levi subgroup of G



Then $\Pi_\psi^A(G)$ is unitary $\iff$ $\Pi_{\psi_M}^A(M)$ is unitary.

Arthur's conjectures

For unipotent Arthur packets, unitarity is known in the following cases:

- G complex classical (Barbasch)
- G quasi-split classical (Arthur)
- G real exceptional (Adams-van Leeuwen-Miller-Vogan)
- G real classical (Barbasch-Ma-Sun-Zhu)
- G complex exceptional of rank ≤ 6 (Atlas calculation)

Only 13 distinguished Arthur packets which are not known to be unitary (in complex E_7, E_8)!

Corollary (Adams-MB-Vogan)

Suppose G is a real form of a simple exceptional group. Then Arthur's unitarity conjecture is true.

The ABV point of view

- Starting point of ABV: should be a relationship

$$\text{rep theory of } G \longleftrightarrow \text{geometry of } \Phi({}^\vee G^\Gamma).$$

- Roughly: character formulas for G should correspond to closure relations on ${}^\vee G$-orbits in $\Phi({}^\vee G^\Gamma)$.
- Problem: all ${}^\vee G$-orbits on $\Phi({}^\vee G^\Gamma)$ are closed!
- Solution in ABV: find some closely related space of parameters with more interesting geometry.

The ABV parameter space

The ABV parameter space is a ${}^\vee G$-eqvt quotient of $\Phi({}^\vee G^\Gamma)$

$$\Phi({}^\vee G^\Gamma) \twoheadrightarrow X({}^\vee G^\Gamma).$$

It has the following properties:

- The induced map

  $$\Phi({}^\vee G^\Gamma)/{}^\vee G \to X({}^\vee G^\Gamma)/{}^\vee G$$

  is a bijection.
- If $\varphi \mapsto x$, the natural inclusion

  $$({}^\vee G)^\varphi \subseteq ({}^\vee G)^x$$

  is a maximal reductive subgroup. In particular, there is an isomorphism of local cmpnt groups

  $$A(\varphi) = ({}^\vee G)^\varphi/({}^\vee G)^\varphi_0 \xrightarrow{\sim} ({}^\vee G)^x/({}^\vee G)^x_0 = A(x).$$

The ABV parameter space

There is also a map

$$X({}^\vee G^\Gamma) \twoheadrightarrow {}^\vee\mathfrak{g} /\!\!/ {}^\vee G \simeq \mathfrak{h}^*/W$$

such that the following diagram commutes

$$\begin{array}{ccc}
\Phi({}^\vee G^\Gamma) & \longrightarrow & X({}^\vee G^\Gamma) \\
\downarrow{\scriptstyle \varphi \mapsto d\varphi(1)} & & \downarrow \\
{}^\vee\mathfrak{g} /\!\!/ {}^\vee G & = & {}^\vee\mathfrak{g} /\!\!/ {}^\vee G \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
\mathfrak{h}^*/W & = & \mathfrak{h}^*/W
\end{array}$$

The ABV parameter space

Fix (once and for all) $\lambda \in \mathfrak{h}^*/W$, and write

$$X_\lambda({}^\vee G^\Gamma) := \text{preimage of } \lambda \text{ under } X({}^\vee G^\Gamma) \to \mathfrak{h}^*/W$$

Proposition (ABV)

$X_\lambda({}^\vee G^\Gamma)$ is a smooth complex algebraic variety with finitely-many ${}^\vee G$-orbits. It is built (in a certain precise sense) out of partial flag varieties.

Example

Let $G = PGL(2,\mathbb{C})$ (unique inner class) and $\lambda = \rho$. Then

$$X_\lambda({}^\vee G^\Gamma) = SL(2,\mathbb{C}) \times_{SO(2,\mathbb{C})} (SL(2,\mathbb{C})/B)$$

Complete geometric parameters

Definition (ABV)

A complete geometric parameter for ${}^\vee G^\Gamma$ is a pair

$$(S, \tau)$$

consisting of

- a ${}^\vee G$-orbit $S \subset X_\lambda({}^\vee G^\Gamma)$, and
- an irreducible ${}^\vee G$-equivariant local system τ on S.

Write $\Xi({}^\vee G^\Gamma)$ for the set of complete geometric parameters.

Complete geometric parameters and constructible sheaves

Each $\xi \in \Xi({}^\vee G^\Gamma)$ determines two different elements of

$$KX_\lambda({}^\vee G^\Gamma) = \text{Grothendieck group of } {}^\vee G\text{-equivariant constructible sheaves on } X_\lambda({}^\vee G^\Gamma).$$

Namely:

- $\mu(\xi)$ = extension by 0 (irreducible constructible sheaf).
- $P(\xi)$ = IC extension (irreducible perverse sheaf).

Lemma

The sets

$$\{\mu(\xi) \mid \xi \in \Xi({}^\vee G^\Gamma)\} \quad \text{and} \quad \{P(\xi) \mid \xi \in \Xi({}^\vee G^\Gamma)\}$$

are two (different) $\mathbb{Z}$-bases for $KX_\lambda({}^\vee G^\Gamma)$.

Complete geometric parameters and representations

Each $\xi \in \Xi({}^\vee G^\Gamma)$ also determines two different elements of

$$K\Pi_\lambda(G) = \text{Grothendieck group of finite-length reps of real forms of } G \text{ of infl char } \lambda$$

Namely:

- $\pi(\xi)$ = irreducible representation parameterized by ξ.
- $M(\xi)$ = associated standard representation.

Lemma

The sets

$$\{\pi(\xi) \mid \xi \in \Xi({}^\vee G^\Gamma)\} \quad \text{and} \quad \{M(\xi) \mid \xi \in \Xi({}^\vee G^\Gamma)\}$$

are two (different) $\mathbb{Z}$-bases for $K\Pi_\lambda(G)$.

Vogan duality

Definition (Vogan duality)

Vogan duality is the perfect pairing

$$\langle \cdot, \cdot \rangle : K\Pi_\lambda(G) \times KX_\lambda({}^\vee G^\Gamma) \to \mathbb{Z}$$

defined by

$$\langle M(\xi), \mu(\xi') \rangle = \delta_{\xi,\xi'}.$$

Theorem (ABV)

Vogan duality satisfies

$$\langle \pi(\xi), P(\xi') \rangle = (-1)^{d(\xi)} \delta_{\xi,\xi'}.$$

Microlocal geometry on the ABV parameter space

Let P be a ${}^\vee G$-equivariant perverse sheaf on $X_\lambda({}^\vee G^\Gamma)$. Then the *characteristic cycle* of P is a $\mathbb{Z}$-linear combination of co-normal bundles for ${}^\vee G$-orbits on $X_\lambda({}^\vee G^\Gamma)$:

$$\mathrm{CC}(P) = \sum_S m_S(P) \cdot T^*_S(X_\lambda).$$

Coefficients $m_S(\bullet)$ are called *microlocal multiplicities*. Each is a $\mathbb{Z}$-linear functional on $KX_\lambda({}^\vee G^\Gamma)$.

Lemma

Let $\xi = (S, \tau) \in \Xi({}^\vee G^\Gamma)$ and let $P = P(\xi)$. Then

- If $m_{S'}(P) \neq 0$, then $S' \subset \overline{S}$.
- $m_S(P) \neq 0$.

Arthur packets from the ABV point of view

Definition (ABV)

Let $\psi \in \Psi({}^\vee G^\Gamma)$, and let $S \subset X_\lambda({}^\vee G^\Gamma)$ be the ${}^\vee G$-orbit of the point in $X_\lambda({}^\vee G^\Gamma)$ corresponding to φ_ψ. Then

$$\Pi^A_\psi(G) := \{\pi(\xi) \mid m_S(P(\xi)) \neq 0\}.$$

By the lemma, $\Pi^A_\psi(G)$ contains the L-packet

$$\Pi^L_{\varphi_\psi}(G) = \{\pi(S,\tau) \mid \tau\},$$

along with (possibly) some representations contained in L-packets attached to ${}^\vee G$-orbits S' with $S \subset \partial S'$.

Langlands functoriality from the ABV point of view

Very general setting for Langlands functoriality:

$H =$ complex connected reductive algebraic group, equipped with inner class of real forms

${}^\vee H^\Gamma =$ E-group of H

Also fix an L-homomorphism

$$\epsilon : {}^\vee H^\Gamma \to {}^\vee G^\Gamma$$

Langlands functoriality

The L-homomorphism induces a continuous map

$$\epsilon : X({}^{\vee}H^{\Gamma}) \to X({}^{\vee}G^{\Gamma})$$

If we fix a semisimple element $\lambda \in {}^{\vee}\mathfrak{h}$, we get a semisimple element $d\epsilon(\lambda) \in {}^{\vee}\mathfrak{g}$. And ϵ restricts to algebraic map

$$\epsilon : X_{\lambda}({}^{\vee}H^{\Gamma}) \to X_{\lambda}({}^{\vee}G^{\Gamma}).$$

Pullback along ϵ is an exact functor, and therefore defines a group homomorphism

$$\epsilon^{*} : KX_{\lambda}({}^{\vee}G^{\Gamma}) \to KX_{\lambda}({}^{\vee}H^{\Gamma}).$$

Taking transpose and applying Vogan duality, we get a group homomorphism (called 'transfer' or 'lifting')

$$\epsilon_{*} : K\Pi'_{\lambda}(H) \to K\Pi_{\lambda}(G).$$

Langlands functoriality from the ABV point of view

It is easy to see how ϵ_* behaves on standard representations.

Proposition

Let $S_H \subset X_\lambda({}^\vee H^\Gamma)$ and let $S_G = {}^\vee G \cdot \epsilon(x_H)$. There is a natural homomorphism

$$\epsilon : A(S_H) \to A(S_G).$$

On the level of standards

$$\epsilon_* M(S_H, \tau_H) = \sum_{\tau_G} \dim \mathrm{Hom}_{A(S_H)}(\tau_H, \epsilon^* \tau_G) \cdot M(S_G, \tau_G).$$

Langlands functoriality and Arthur packets

Much more subtle question:

Question 1

How do Arthur packets behave under ϵ_*?

This is more or less equivalent to

Question 2

How do characteristic cycles behave under ϵ^*?

This is a very difficult question. Not much can be said at this level of generality...

Pseudo-endoscopic groups

Here is a general setting in which more can be said.

Definition (Adams-MB-Vogan)

We say that (H, ϵ) is a *pseudo-endoscopic group* if there exists a finite-order element $s \in {}^\vee G$ such that ${}^\vee H$ is the identity component of the centralizer of s.

Remark

If we require that s commutes with *all of* ${}^\vee H^\Gamma$, then we recover the Langlands-Shelstad notion of an endoscopic group. In this case, ϵ_* coincides (on stable chars) with endocopic lifting.

Remark

For any $y \in {}^\vee H^\Gamma \setminus {}^\vee H$, we have $sys^{-1} = zy$ for a fixed element $z \in Z({}^\vee H)$ taken to its inverse by action of Γ. This defines a 1-cocycle $\Gamma \to Z({}^\vee H)$. Trivial iff (H, ϵ) is endoscopic.

Pseudo-endoscopic groups

We have proved some general theorems about pseudo-endoscopic lifting. But for the purposes of this talk, we will focus our attention on a particular class of examples, namely *rational Levi subgroups*.

Levi subgroups

Let M be a Levi subgroup of G such that the root datum of M is stable under Γ.

It is easy to find an L-homomorphism

$$\epsilon : {}^\vee M^\Gamma \hookrightarrow {}^\vee G^\Gamma$$

from an E-group of M (if G is simply connected, we can take ${}^\vee M^\Gamma$ to be an L-group. In general we cannot).

Clearly, (M, ϵ) is pseudo-endoscopic, usually not endoscopic.

Example

Let $G = PGL(2, \mathbb{R})$ and let M be the compact Cartan subgroup. Then there is no ϵ for which (M, ϵ) is an endoscopic group.

Arthur packets and pseudo-endoscopic lifting

Let $\psi_M \in \Psi({}^\vee M^\Gamma)$ and let $\psi_G = \epsilon \circ \psi_M \in \Psi({}^\vee G^\Gamma)$.

Theorem 1 (Adams-MB-Vogan)

$$\Pi^A_{\psi_G}(G) \subseteq [\epsilon_* \Pi^A_{\psi_M}(M)].$$

(RHS interpreted as set of irr constituents).

Unraveling definitions, we see that this equivalent to the following:

Theorem 1' (Adams-MB-Vogan)

For any irr eqvt perverse sheaf P on $X_\lambda({}^\vee G^\Gamma)$, we have

$$m_{S_G}(P) \neq 0 \implies m_{S_M}(\epsilon^* P) \neq 0.$$

(in fact, we prove $\iff$)

Idea of proof

Let

$$Y_\lambda := {}^\vee G \times_{{}^\vee M} X_\lambda({}^\vee M^\Gamma)$$

There is a diagram

$$\begin{array}{ccc} X_\lambda({}^\vee M^\Gamma) & & \\ \downarrow i & \overset{\epsilon}{\searrow} & \\ Y_\lambda & \xrightarrow{\;p\;} & X_\lambda({}^\vee G^\Gamma) \end{array}$$

where

- i is a closed immersion, and
- p is a smooth fiber bundle.

Idea of proof

Since (M, ϵ) is pseudo-endoscopic

- there is a finite-order automorphism

$$\sigma : Y_\lambda \to Y_\lambda, \qquad \sigma([g, x]) = [\mathrm{Ad}(s)g, x]$$

- $X_\lambda({}^\vee M^\Gamma) = (Y_\lambda)^\sigma$.

This equality allows for application of a **Lefschetz fixed-point theorem**. Can be used to calculate microlocal traces.

Cohomological induction

Fix a *complex* parabolic subalgebra $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{u} \subset \mathfrak{g}$. Then there is a group homomorphism

$$R^{\mathfrak{g}}_{\mathfrak{q}} : K\Pi'_{\lambda}(M) \to K\Pi_{\lambda}(G)$$

called *cohomological induction*. Two extreme cases:

- If $\mathfrak{q}$ is real, then $R^{\mathfrak{g}}_{\mathfrak{q}}$ is ordinary parabolic induction.
- If $\mathfrak{q}$ is θ-stable, then $R^{\mathfrak{g}}_{\mathfrak{q}}$ is a Zuckerman functor.

Remark

$R^{\mathfrak{g}}_{\mathfrak{q}}$ is known to preserve unitarity under some conditions on the inducing module (which are vacuous if $\mathfrak{q}$ is real).

Question

What is the relationship between ϵ_* and $R^{\mathfrak{g}}_{\mathfrak{q}}$?

Cohomological induction and pseudo-endoscopic lifting

(Unsatisfying) answer:

Lemma (Adams-MB-Vogan)

There is an element w in the integral Weyl group of λ such that

$$\epsilon_* = T_w \circ R_{\mathfrak{q}}^{\mathfrak{g}} : K\Pi'_\lambda(M) \to K\Pi_\lambda(G)$$

where T_w denotes the coherent continuation action of w.

On the level of Arthur packets, there is a more striking relationship:

Theorem 2 (Adams-MB-Vogan)

Suppose ψ_M is a unipotent Arthur parameter. Then

$$[\epsilon_* \Pi^A_{\psi_M}(M)] \subseteq [R_{\mathfrak{q}}^{\mathfrak{g}} \Pi^A_{\psi_M}(M)].$$

Proof of main result

Let $\psi_G \in \Psi({}^\vee G^\Gamma)$.

Can always find an E-group ${}^\vee M^\Gamma \subset {}^\vee G^\Gamma$ (not necessarily unique!) such that ψ_G factors through a distinguished Arthur parameter $\psi_M : W_\mathbb{R} \times SL(2,\mathbb{C}) \to {}^\vee M^\Gamma$.

By Theorem 1:

$$\Pi^A_{\psi_G}(G) \subseteq [\epsilon_* \Pi^A_{\psi_M}(M)].$$

By Theorem 2:

$$[\epsilon_* \Pi^A_{\psi_M}(M)] \subseteq [R^{\mathfrak{g}}_{\mathfrak{q}} \Pi^A_{\psi_M}(M)].$$

Conclusion:

$$\Pi^A_{\psi_G}(G) \subseteq [R^{\mathfrak{g}}_{\mathfrak{q}} \Pi^A_{\psi_M}(M)]$$

RHS consits of unitary reps.