A relative trace formula approach to the stable trace formula on the unitary group

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Jacquet-Rallis relative trace formula

- E/F quadratic extension of number fields, $\eta : \mathbb{A}_F^\times \to \{\pm 1\}$ quadratic character.
- $G = \mathrm{Res}_{E/F}(\mathsf{GL}_n \times \mathsf{GL}_{n+1})$.

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- Subgroups of G:

$$H_1 = \{(x, \begin{pmatrix} x & \\ & 1 \end{pmatrix}) \mid x \in \mathrm{GL}_{n,E}\}, \quad H_2 = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n+1,F}\,.$$

- $f \in \mathcal{S}(G(\mathbb{A}_F))$ a test function, $K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$ automorphic kernel function.

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Studies on Jacquet-Rallis RTF:

- Smooth Transfer: Zhang, Xue.
- Fundamental lemma: Yun–Gordan, Beuzart-Plessis, Zhang.
- Regularization: Zydor.
- Singular terms: Chaudoaurd–Zydor, Beuzart-Plessis–Chaudouard–Zydor, Beuzart-Plessis–Chaudouard.

RTF on the Lie algebra

$f \in \mathcal{S}(\mathfrak{gl}_{n+1}(\mathbb{A}_F))$, $h \in [\mathrm{GL}_n]$

$$K_f(h) = \sum_{X \in \mathfrak{gl}_{n+1}(F)} f(h^{-1}Xh).$$

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$\mathcal{A} := \mathfrak{gl}_{n+1} / \mathrm{GL}_n$ the GIT quotient, $a \in \mathcal{A}(F)$.

$$K_{f,a}(h) := \sum_{X \mapsto a} f(h^{-1}Xh), \quad I_a(f) := \int_{[\mathrm{GL}_n]} K_{f,a}(h)\eta(h)dh.$$

Geometric expansion:

$$I(f) = \sum_{a \in \mathcal{A}(F)} I_a(f)$$

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$\mathcal{F} := \{$semistandard psgps of GL_{n+1} s.t. $P \cap \mathsf{GL}_n$ is standard$\}$.

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$$K_a^T(f) := \sum_{P \in \mathcal{F}} \varepsilon_P \sum_{\gamma \in P_n(F) \backslash \mathrm{GL}_n(F)} \widehat{\tau}_P(H_P(\gamma h) - T_P) K_{f,P,a}(\gamma h).$$

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Zydor: absolutely convergent, exponential polynomial in T, pure polynomial term is a constant $=: I_a(f)$.

Geometric terms

- $X \in \mathfrak{gl}_{n+1}(F)$ is (relatively) regular if its stabilizer if trivial.
- X is (relatively) regular semisimple, if it is regular and the orbit of X is closed.
- $a \in \mathcal{A}(F)$ is regular semisimple if it is the image of a regular semisimple element. In this case, choose any $X \mapsto a$

$$I_a(f) = \mathrm{Orb}(X, f) = \int_{\mathrm{GL}_n(\mathbb{A})} f(h^{-1}Xh)\eta(h)dh.$$

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- Q: Can we describe $I_a(f)$ for general a?
  Chaudouard–Zydor: global semisimple descent, reduce to study $I_a(f)$ for nilpotent a.
- Today: study the "regular part" of $I_a(f)$ for any $a \in \mathcal{A}(F)$.

Regular orbit

- $X \in \mathfrak{gl}_{n+1}(F)$ is (relatively) regular if its stabilizer if trivial.
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- One special regular orbit: X is called $+$-regular if $e_{n+1}, Xe_{n+1}, \cdots, X^n e_{n+1}$ are linearly independent.
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Example

- a r.s.s, $+$-regular orbit $=$ the unique orbit above a.
- $a = 0, X =$ the principal Jordan block (1 above the diagonal)

Regular contribution

Theorem (L. 24)

If f is supported in $+$-regular open subset. For $a \in \mathcal{A}(F)$, choose $X \mapsto a$, then the integral

$$\mathrm{Orb}(X, f, s) = \int_{\mathsf{GL}_n(\mathbb{A})} f(h^{-1}Xh)\eta(h)|\det h|^s dh$$

is convergent when $\mathrm{Re}(s) \ll 0$, and has analytic continuation to $s = 0$. Moreover

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We have a more general theorem for f supported in the regular subset.

$$I_a(f) = \sum_i \mathrm{Orb}(X_i, f, 0),$$

the sum runs through regular orbits above a.

Sketch of the proof

- Zydor:

$$I_a(f,s) := \int_{[\mathrm{GL}_n]}^{\mathrm{reg}} K_{f,a}(x)\eta(x)|x|^s dx,$$

defined as constant term of the exponential polynomial

$$I_a^T(f,s) := \int_{[\mathrm{GL}_n]} K_{f,a}^T(x)\eta(x)|x|^s dx.$$

- Show that under the assumption

$$\lim_{T\to\infty} I_a^T(f,s) = I(f,s) = \mathrm{Orb}(X,f,s)$$

when $\mathrm{Re}(s) \ll 0$, by computing explicitly the exponents.

Local theory

The global $\mathrm{Orb}(X, f, s)$ is Eulerian, leading to the study of local integrals.

- E/F quadratic ext'n of local fields, $\eta : F^\times \to \{\pm 1\}$ quadratic char.
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We consider

$$\mathrm{Orb}(X, f, s) := \int_{\mathrm{GL}_n(F)} f(h^{-1}Xh)\eta(h)|\det h|^s dh.$$

Fact:

- $\mathrm{Orb}(X, f, s)$ is convergent for $\mathrm{Re}(s) < 1 - \frac{1}{n}$, and has meromorphic continuation to $\mathbb{C}$.
- Poles are controlled by an abelian L-function: $\exists L_a(s) =$ g.c.d

$$\mathrm{Orb}^\natural(X, f, s) := \mathrm{Orb}(X, f, s)/L_a(s) \text{ is entire.}$$

Local transfer

Example

- a r.s.s, $L_a(s) = 1$.
- $a = 0$, $L_a(s) = L(-s, \eta)L(-2s-1, \eta^2) \cdots L(-ns-n+1, \eta^n)$.

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Orbital integral can be compared to orbital integral on the unitary group.

- $\mathcal{H}$ isom. class of n-dim Hermitian spaces.
- $V \in \mathcal{H}, \mathfrak{u}^V := \{X \in \mathrm{End}_E(V) \mid X \text{ self-adjoint}\}$, $U(V)$ acts on $\mathfrak{u}^V$ by conjugation.
- $V \in \mathcal{H}$, $V' := V \oplus E$. Consider $U(V)$ action on $\mathfrak{u}^{V'}$.
- The GIT quotient $\mathfrak{u}^{V'}/U(V)$ can be identified with $\mathcal{A}$.
- We have similar notion of regular and regular semisimple element.

Local transfer contd.

Fact: for any $a \in \mathcal{A}(F)$ r.s.s., there exists a unique $V \in \mathcal{H}$ such that $\mathfrak{u}_a^{V'}(F)$ (the fiber of a) $\neq \varnothing$.

Local transfer contd.

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$f \in \mathcal{S}(\mathfrak{gl}_{n+1}(F))$ and $f^V \in \mathcal{S}(\mathfrak{u}^{V'})(F)$. f and f^V are transfer if for any $a \in \mathcal{A}(F)$ r.s.s., we have

$$\mathrm{Orb}(X, f)\omega(X) = Orb(X^V, f^V),$$

where

- $V \in \mathcal{H}$ such that $\mathfrak{u}_a^{V'}(F) \neq \varnothing$.
- $\omega(X)$ is the transfer factor (s.t. Orb only depends on a but not X).
- $X^V \mapsto a$ and $\mathrm{Orb}(X^V, f^V) = \int_{U(V)(F)} f^V(h^{-1}X^V h)dh$.

Local singular transfer

Our local theorem states that when f and $(f^V)_{V \in \mathcal{H}}$ matches, then for any $a \in \mathcal{A}(F)$, the regular orbital integral also matches with the semisimple orbital integral.

Theorem (L. 24)

For matching f and $(f^V)_{V \in \mathcal{H}}$, we have

$$\mathrm{Orb}^{\natural}(X, f, 0)\omega(X) = \sum_{(Y,V)} c_Y \mathrm{Orb}(Y, f^V),$$

where the sum runs through (Y, V), s.t. Y is a semisimple orbits in $\mathfrak{u}_a^{V'}$.

regular orbital integral on GL_n = (stable) semisimple orbital integral on U_n.

Sketch of the proof

- Zhang: Relative semisimple descent:
  regular orb. = r.s.s orb. × reg. unipotent orb.
- Reduce the the r.s.s and the unipotent case.
- r.s.s. case follows from the definition.
- Beuzart-Plessis: regular unipotent case using Fourier transform commutes with transfer (due to Zhang, Xue).

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Local singular transfer revisited

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Regular orbital integral on $\mathfrak{gl}_n \times F^n \leftrightarrow$ Stable orbital integral on $\mathfrak{u}^V$.

Globalization

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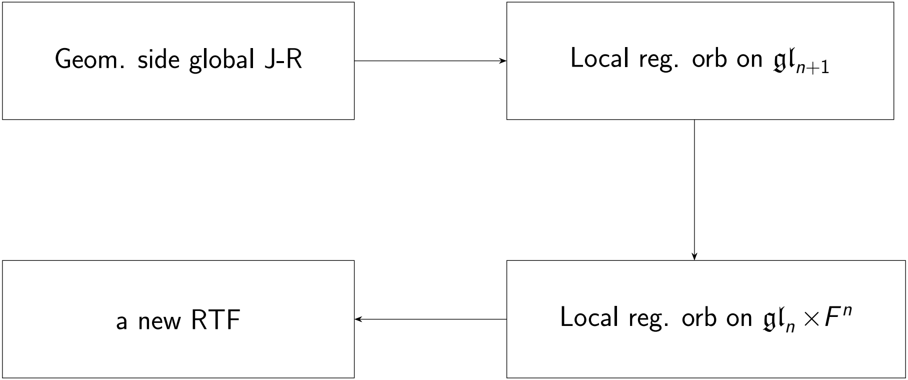
- $f \in \mathcal{S}(\mathrm{GL}_n(\mathbb{A}_E)), \Phi \in \mathcal{S}(\mathbb{A}^n)$.

Consider

$$I(f \otimes \Phi, s) = \int_{[\mathrm{GL}_n]} \int_{[\mathrm{GL}_n]} K_f(x, y)\Theta(x, \Phi)\eta(x)^n \eta(y)^{n+1}|x|^s dxdy.$$

where $\Theta(x, \Phi) = \sum_{v \in F^n} \Phi(xv)$.

Summary


Geom. side global J-R
Local reg. orb on $\mathfrak{gl}_{n+1}$
Local reg. orb on $\mathfrak{gl}_n \times F^n$
a new RTF

Geometric expansion

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Geometrically, $\mathrm{GL}_{n,E} \times F^n / \mathrm{GL}_n \times \mathrm{GL}_n$, the action is given by

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Facts:

- Infinitesimally looks like $(\mathfrak{gl}_n \times F^n) / \mathrm{GL}_n$,
- $\exists$ canonical identification

$$\mathrm{GL}_{n,E} \times F^n / \mathrm{GL}_n \times \mathrm{GL}_n \cong U(V) /_{\mathrm{conj}} U(V)$$

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Upshot: it can be compared to the STF on the unitary group!

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$$I^T(f\otimes\Phi, s) := \int_{[\mathrm{GL}_n]}\int_{[\mathrm{GL}_n]} K^T_{f\otimes\Phi}(x,y)\eta(x)^n\eta(y)^{n+1}|\det y|^s dxdy.$$

Theorem (Chen-L.-Zhang)

Absolutely convergent, exponential polynomial in T, pure polynomial term is constant ($:= I(f\otimes\Phi, s)$) whenever $s \neq 0, 1$, meromorphic in s.

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Example

- $P = \mathrm{GL}_{n+1}$, the usual Θ function.
- P lower triangular, Levi $\mathrm{GL}_n \times \mathrm{GL}_1$, ${}_P\Theta(x, \Phi) = \Phi(0)$.
- P upper triangular, Levi $\mathrm{GL}_n \times \mathrm{GL}_1$, ${}_P\Theta(x, \Phi) = |x|^{-1}\widehat{\Phi}(0)$.

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Partial Θ-function also appears in a joint work with Boisseau and Xue on the GGP conjecture for Fourier-Jacobi periods.

Test functions

$f \otimes \Phi = \otimes_v (f_v \otimes \Phi_v) \in \mathcal{S}(\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}_F^n)$ is a nice test function, if

- $\exists$ a non-Arch v_1 s.t. f_{v_1} is truncated matrix coefficient of supercuspidal representation.
- $\exists$ a non-Arch $v_2 \neq v_1$ split in E s.t $\mathrm{supp}(f_{v_2}) \subset$ the elliptic locus.

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- $\exists$ a non-Arch $v_2 \neq v_1$ split in E s.t $\mathrm{supp}(f_{v_2}) \subset$ the elliptic locus.

Let V_0 be the split Hermitian space, $U := U(V_0)$.

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Comparison

Theorem (L. 2025)

Let $f \otimes \Phi$ and f^{V_0} be matching nice test functions. Then

- *The distribution $I(f \otimes \Phi, s)$ is holomorphic at $s = 0$.*
- *We have $2I(f \otimes \Phi, 0) = S(f^{V_0})$.*

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Some remarks:

- The distribution $I(f \otimes \Phi, s)$ on $\mathcal{S}(\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}_F^n)$ is stable in the sense that

  $$s \neq 0, 1, g \in \mathrm{GL}_n(\mathbb{A}_F) \implies I(R(g)(f \otimes \Phi), s) = \eta(g)|g|^s I(f \otimes \Phi, s).$$

  However, it has pole at $s = 0$.
- One expects there is some way to stabilize the GL_n and compare full trace formulas.

Table of content

Diagonal cycle

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- E/F CM ext'n of number field.
- V n-dim'l Herm space, signature $(n-1,1),(n,0),\cdots,(n,0)$.
- $X := \mathrm{Sh}_H$, $\mathfrak{X}$ integral model over $\mathcal{O}_F$, abs. dim $= n$.
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Conjecture(Chen-L.-Zhang): π tempered cohomological,

$$\langle \Delta_\pi, \Delta_\pi, \widehat{\omega} \rangle \sim L'(1, \pi, \mathrm{Ad}),$$

$\langle \cdot, \cdot \rangle$ denotes the Arakelov intersection pairing, $\widehat{\omega}$ is the (metrized) Hodge bundle.

Diagonal cycle cont'd

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The conjecture holds when $n = 2$.

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Theorem (Chen-L.-Zhang)

The conjecture holds when $n = 2$.

- $n = 1$: Faltings height of CM abelian variety (average Colmez conj.)
- Relative Langlands duality(Ben-Zvi–Sakellaridis-Venkatesh) provides a general framework for periods (automorphic/geometric/arithmetic).

Proof

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Proposed by Zhang, and success in the (p-adic) arithmetic GGP conjecture for Bessel case (Disegni–Zhang).

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Main proposition: when f and $(f' \otimes \Phi)$ are transfer, then

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- π-part of LHS: the intersection pairing we're interested in.
- π-part of RHS: $L'(0, \pi, \mathrm{Ad})$

Stable character

- E/F local, ϕ a tempered L-parameter of U.
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For $f_v \in \mathcal{S}(\mathrm{GL}_n(E))$ and $\Phi \in \mathcal{S}(F^n)$, we put

$$I_{\Pi_v}(f_v \otimes \Phi_v, s) = \sum_{W \in \mathcal{B}_{\Pi_v}} Z_n(R(f_v)W, \Phi_v, s)\beta(\overline{W}),$$

and a normalized version

$$I^\natural_{\Pi_v}(f_v \otimes \Phi_v, s) = \sum_{W \in \mathcal{B}_{\Pi_v}} Z^\natural_n(R(f_v)W, \Phi_v, s)\beta^\natural(\overline{W}),$$

Z local Flicker-Rallis zeta integral, β local Flicker-Rallis period.

Spectral comparison

Theorem (L. 25)

There exists $C(\phi) \in \mathbb{C}$ such that

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holds for all matching of function (f^V) and $f \otimes \Phi$.

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Proof: globalization and using the global comparison

Thank you for your attention!!