Periods in algebraic geometry and automorphic forms

Analytic cocycles and explicit class field theory: introductory survey

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The Kronecker-Weber Theorem

The prototypical theorem:

Theorem (Kronecker-Weber): All abelian extensions of $\mathbb{Q}$ are contained in a field generated by n-th roots of unity.

In other words, these extensions can be obtained by adjoining to $\mathbb{Q}$ the values of the (transcendental) function $e^{2\pi iz}$ at rational arguments.

This lovely 19th Century result leads to a concrete formulation of class field theory for $\mathbb{Q}$, which is also instrumental for the extension to other number fields.

The theory of complex multiplication

Let K be a quadratic imaginary field.

Theorem. Almost all abelian extensions of K are contained in a field generated by *singular moduli* attached to K: values of the modular function $j(z)$ at arguments in K.

The function $j(z)$ plays the role of the exponential function $e^{2\pi iz}$ in extending the Kronecker-Weber theorem to quadratic imaginary base fields.

Question: What about other number fields?

Hilbert's Twelfth Problem

David Hilbert championed the question of how one might generate class fields of more general base fields from explicit values of appropriate transcendental functions.

It was formulated as the *twelfth problem* in his celebrated address at the 1900 ICM in Paris.

What functions play the role, for general number fields K, of the trigonometric functions for $\mathbb{Q}$ and of modular functions for quadratic imaginary fields?

Hilbert's approach

Hilbert's idea was to approach explicit class field theory through the study of modular functions of several complex variables (*Hilbert modular forms*) and their values at special points corresponding to moduli of *abelian varieties with complex multiplication*.

The idea is to generate class fields by the values of modular functions on higher dimensional Shimura varieties – most notably, Hilbert modular varieties – at CM points.

The work of Weil, Shimura-Taniyama

The general program initiated by Hilbert and his students (Blumenthal, and others) was brought to fruition by Weil, and Shimura and Taniyama in their 1960's treatise on complex multiplication.

Shimura and Taniyama's extension of the theory of complex multiplication applies to a broad swathe of number fields – the so called CM fields.

It is unconditional, and very much in the spirit of the original theory of complex multiplication.

A critique of complex multiplication

" Within the framework of abelian extensions, we have also made no progress towards the generalization of Kronecker's "dreams of youth", the generation of class fields whose existence is known to us, by values of analytic functions. Although we were able, without great difficulty, to complete the unfinished work of Kronecker and obtain the solution of this problem by means of the theory of complex multiplication in the case of imaginary quadratic fields, the key to the general problem which Hilbert regarded as one of the most important of modern mathematics still eludes us [...] one cannot fail to obtain interesting results by examining these questions more closely."

André Weil, *L'avenir des mathématiques*. (1950)

What about base fields that are not CM fields? e.g., $K =$ a real quadratic field?

Original French version

" Dans le cadre même des extensions abéliennes, nous n'avons non plus fait aucun progrès vers la généralisation des théorèmes du "rêve de jeunesse de Kronecker", l'engendrement des corps de classes dont l'existence nous est connue, par des valeurs de fonctions analytiques. Si l'on a pu, sans grande difficulté, compléter l'oeuvre inachevée de Kronecker et achever de résoudre ce problème, au moyen de la multiplication complexe, dans le cas des corps quadratiques imaginaires, la clef du problème général que Hilbert regardait comme l'un des plus importants des mathématiques modernes, nous échappe encore, [...] on ne peut manquer d'obtenir des résultats intéressants en examinant ces questions de plus près."

André Weil, *L'avenir des mathématiques* (1950).

Stark's Conjecture

In the 1960's, Stark formulated his important conjectures, representing the first serious proposal towards explicit class field theory beyond the work of Shimura and Taniyama.

Kronecker's Limit formula: If $\mathfrak{a}$ is an ideal of a quadratic imaginary field, then

$$L'(K, \mathfrak{a}, 0) = \log(\sqrt{y}|\eta(\tau_{\mathfrak{a}})|^2), \qquad \tau_{\mathfrak{a}} = x + iy$$

where η is the Dedekind eta-function.

Stark's conjecture. The leading terms of abelian L-series with simple zeroes at $s = 0$ can be expressed in terms of logarithms of units in abelian extensions of the ground field.

Critique of Stark's conjectures

a) Stark's conjectures remain largely open and inaccessible so far. They make no proposal for what might arise in the "right hand side" of a putative generalisation of the Kronecker limit formula for other base fields.

b) The settings where abelian L-series $L(K, \chi, s)$ admit simple zeroes at $s = 0$ are somewhat restricted. Namely, K, χ must be either

- *totally real*, with χ odd at all but one real place of K;
- or ATR (*almost totally real*, i.e., have single complex place).

Stark's conjecture says little, even conjecturally, about base fields with more than one complex place. It leaves out CM fields!

CM theory and Stark's conjecture

The theory of complex multiplication and Stark's conjectures are somewhat complementary, *except* for quadratic imaginary fields which motivated the latter. For instance, we do not know:

• what is the Kronecker limit formula for CM fields? Can one obtain units in abelian extensions from values of (Hilbert) modular functions? (Cf. de Shalit, Eyal Goren's PhD thesis.)

• What plays the role of $\eta(z)$ for (say) real quadratic fields, or more general base fields?

• The theory of complex multiplication yields a rich panoply of algebraic invariants defined over class fields – *Heegner points* on modular elliptic curves, and *singular moduli*. The framework of the Stark conjecture is not flexible enough to suggest clear analogues.

The cubic cocycle of Bergeron, Charollois, and Garcia

In a remarkable recent work, the authors construct a one-cocycle

$$J_{\mathrm{BCG}} \in Z^1(\mathsf{SL}_3(\mathbb{Z}), \mathcal{A}^\times),$$

where $\mathcal{A}^\times$ is the multiplicative group of functions that are holomorphic (on *suitable open subsets* of $\mathbb{P}_2(\mathbb{C})$). It is obtained through an appealing classical process of Eisenstein summation.

Conjecture. Let T be a torus in $\mathsf{SL}_3(\mathbb{Q})$ with $T(\mathbb{Q}) = K_1^\times$, and $T(\mathbb{Z}) = \mathcal{O}_{K,1}^\times$, where K is a cubic field with one complex place. Let $v \in \mathbb{P}_2(\mathbb{C})$ be a complex eigenvector for T. The *value*

$$J_{\mathrm{BCG}}[T] := \mathrm{eval}_v(\mathsf{res}_{T(\mathbb{Z})}(J_{\mathrm{BCG}})) \in H^1(\mathcal{O}_{K,1}^\times, \mathbb{C}^\times) = \mathbb{C}^\times$$

is a *unit* in the narrow Hilbert class field of K.

This conjecture is supported by an (unconditional) Kronecker limit formula relating $|J_{\mathrm{BCG}}[T]|$ to derivatives of partial zeta functions.

An earlier antecedent: quadratic ATR extensions

K = quadratic ATR extension of a totally real field F of degree d.

One can construct a cocycle $J_{eis} \in Z^{d-1}(\mathrm{SL}_2(\mathcal{O}_F), \mathcal{A}^\times(\mathcal{H}))$ from periods of Hilbert modular Eisenstein series.

Conjecture (Charollois, D, 2006). Let T be a torus in $\mathrm{SL}_2(\mathcal{O}_F)$ with $T(\mathbb{Q}) = K_1^\times$, and $T(\mathbb{Z}) = \mathcal{O}_{K,1}^\times$. Let $\tau \in \mathcal{H}$ be a fixed point for $T((K \otimes_w \mathbb{R}))$. The *value* $J_{\mathrm{eis}}[T]$, defined by

$$\mathrm{eval}_\tau(\mathrm{res}_{T(\mathbb{Z})}(J_{\mathrm{eis}})) \in H^{d-1}(\mathcal{O}_{K,1}^\times, \mathbb{C}^\times) = H^{d-1}(\mathbb{Z}^{d-1}, \mathbb{C}^\times) = \mathbb{C}^\times$$

is a *unit* in a narrow (relative) Hilbert class field for K/F.

As with the GL_3-cocycle of Bergeron, Charollois and Garcia, this conjecture is supported by an (unconditional) Kronecker limit formula relating $|J_{\mathrm{eis}}[T]|$ to derivatives of partial zeta functions.

Stark-Heegner points

Replacing the Hilbert modular Eisenstein series on F by a weight two cusp form (associated, say, to an elliptic curve E over F), one obtains a class

$$J_E \in Z^{d-1}(\mathrm{SL}_2(\mathcal{O}_F), \mathcal{A}(\mathcal{H})/\Lambda_E).$$

A conjecture of Oda predicts that $\mathbb{C}/\Lambda_E \sim E(\mathbb{C})$.

Conjecture (Logan, D, 2007). Let T be a torus as in the previous slide. The *value* $J_{\mathrm{E}}[T]$, defined by

$$\begin{aligned} \mathrm{eval}_\tau(\mathrm{res}_{T(\mathbb{Z})}(J_E)) &\in H^{d-1}(\mathcal{O}_{K,1}^\times, \mathbb{C}/\Lambda_E) \\ &= H^{d-1}(\mathbb{Z}^{d-1}, \mathbb{C}/\Lambda_E) = \mathbb{C}/\Lambda_E = E(\mathbb{C}) \end{aligned}$$

is defined over the narrow (relative) Hilbert class field for K/F.

Some questions

The replacement of J_{eis} by J_E suggests a more general framework for explicit class field theory going beyond Stark's conjecture, more reminiscent of CM theory.

Question 1: Do the GL_n Eisenstein cocycles of Bergeron, Charollois, and Garcia admit similar cuspidal/elliptic variants?

Question 2: Are there *meromorphic* variants of these cocycles, whose values would give class invariants with non-trivial factorisations rather than just Stark units?

Meromorphic cocycles

The quest for meromorphic cocycles is motivated by the dictionary

Modular function	Modular cocycle	Relation with L-series
Modular units/ Eisenstein series	Eisenstein cocycles $J_{\mathrm{BCG}}, J_{\mathrm{eis}}$	Kronecker limit formula
Modular parametrisations	Elliptic cocycle J_E	Gross-Zagier formula
Meromorphic modular functions	Meromorphic cocycles ???	N/A

The construction of meromorphic cocycles in the Archimedean setting remains somewhat speculative.

So far, the best evidence for the existence of meromorphic cocycles comes from non-archimedean variants.

Ihara cocycles

• Let $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$ = Ihara group.

• $\mathcal{H}_p = \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$ = Drinfeld upper half plane.

• A rigid analytic/meromorphic cocycle, possibly modulo periods, is a one-cocycle in

$$Z^1(\Gamma, \mathcal{A}^\times), \quad \text{or} \quad Z^1(\Gamma, \mathcal{M}^\times), \quad \text{or} \quad Z^1(\Gamma, \mathcal{A}^\times/q^{\mathbb{Z}}).$$

• A rigid cocycle can be evaluated at any RM point $\tau \in \mathcal{H}_p \cap K$ for K a real quadratic field, for which $\Gamma_\tau := \mathrm{Stab}_\Gamma(\tau) \sim \gamma_\tau^{\mathbb{Z}}$.

$$J[\tau] := J(\gamma_\tau)(\tau) \quad \in \quad \mathbb{C}_p^\times, \quad \mathbb{C}_p \cup \{\infty\}, \quad \text{or } \mathbb{C}_p^\times/q^{\mathbb{Z}}.$$

This quantity is called an *RM value*; it is generally conjectured to be an algebraic invariant defined over a ring class field H_τ of the associated real quadratic field $K = \mathbb{Q}(\tau)$.

RM theory via Ihara cocycles

There is a coherent, still *largely conjectural* picture, where RM values resonate with analagous quantities in CM theory.

- There is an *Eisenstein cocycle* $J_{\rm eis} \in Z^1(\Gamma, \mathcal{A}^\times/p^{\mathbb{Z}})$ whose RM values yield Gross-Stark p-units

$$J_{\rm eis}[\tau] \in \mathcal{O}_{H_\tau}[1/p]^\times.$$

- For any elliptic curve $E/\mathbb{Q}$ of conductor p, there is a rigid *elliptic cocycle* $J_E \in Z^1(\Gamma, \mathcal{A}^\times/q_E^{\mathbb{Z}})$, for which

$$J_E[\tau] \stackrel{?}{\in} E(H_\tau).$$

- For any RM point $\tau \in \mathcal{H}_p$, there is a rigid meromorphic cocycle $J_\tau \in Z^1(\Gamma, \mathcal{M}^\times)$ for which $J_\tau(\gamma)$ has zeroes and poles concentrated at $\Gamma\tau$.

$$J_{\tau_1}[\tau_2] \stackrel{?}{\in} (H_{\tau_1} H_{\tau_2})^\times \;\; \sim \;\; \text{“difference of singular moduli”}.$$

An explicit formula for $J_p(\tau_1, \tau_2) := \mathrm{Norm}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}(J_{\tau_1}[\tau_2])$

Assume $\tau_j \in \mathbb{Q}(\sqrt{D_j})$, $\gcd(D_1, D_2) = 1$, $p \nmid D_1 D_2$.

Let $\tau \mapsto \tau'$ be the natural involution sending $\sqrt{D_j}$ to $-\sqrt{D_j}$.

Let γ_j be the geodesic on $\mathcal{H}$ joining τ_j to τ_j' ($j = 1, 2$).

Topological intersection on $\mathcal{H}$: $(\gamma_1 \cdot \gamma_2) \in \{-1, 0, 1\}$.

The *local multiplicative Green's function* is given by

$$g(\tau_1, \tau_2) = \left\{ \frac{(\tau_1 - \tau_2)(\tau_1' - \tau_2')}{(\tau_1 - \tau_1')(\tau_2 - \tau_2')} \right\}^{(\gamma_1 \cdot \gamma_2)} \quad \in \quad \mathbb{Q}(\sqrt{D_1 D_2}).$$

This function is a point-pair invariant:

$$g(\alpha\tau_1, \alpha\tau_2) = g(\tau_1, \tau_2), \qquad \text{for all } \alpha \in \mathrm{SL}_2(\mathbb{Q}).$$

The global Green's function

The group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ acts on the real quadratic elements by Mobius transformations, and on $M_2(\mathbb{Z})$ by left and right multiplication.

$$\Gamma_1 := \mathrm{Stab}_\Gamma(\tau_1), \qquad \Gamma_2 := \mathrm{Stab}_\Gamma(\tau_2).$$

$$G_m(\tau_1, \tau_2) = \prod_{\substack{\alpha \in \Gamma_1 \backslash M_2(\mathbb{Z}) / \Gamma_2 \\ \det(\alpha) = m}} g(\tau_1, \alpha\tau_2) \quad \in \quad \mathbb{Q}(\sqrt{D_1 D_2}).$$

This quantity is a finite product of local Green's functions, and belongs to the quadratic field $\mathbb{Q}(\sqrt{D_1 D_2})$.

Real quadratic singular moduli

Let p satisfy $(D_1/p) = (D_2/p) = -1$ hence $(D_1 D_2/p) = 1$.

$$g(\tau_1, \tau_2), \quad G_m(\tau_1, \tau_2) \in \mathbb{Q}_p.$$

Suppose first that $p = 2, 3, 5, 7$, or 13, i.e., the modular curve $X_0(p)$ has genus zero.

Theorem (D, Vonk)

- *The limit*

$$J_{m,p}(\tau_1, \tau_2) := \lim_{n\to\infty} G_{mp^n}(\tau_1, \tau_2)$$

  exists in $\mathbb{Q}_p$.
- *It belongs to* $H_{\tau_1} H_{\tau_2}$*, up to torsion.*
- *Furthermore,* $J_{1,p}(\tau_1, \tau_2)$ *exhibits the same factorisation patterns as* $j(\tau_1) - j(\tau_2)$ *in the theorem of Gross and Zagier (when* τ_1, τ_2 *are imaginary rather than real quadratic).*

The principle of the proof

The quantity $\log_p J_{m,p}(\tau_1, \tau_2)$ arises in the m-th fourier coefficient of a modular form: the ordinary projection of the diagonal restriction of a first-order p-adic deformation of a Hilbert modular theta series of parallel weight one.

The logarithms of algebraic numbers in $H_1 H_2$ emerge naturally from the **p-adic deformation theory** of weight one Hilbert modular forms and of their **associated Galois representations**. This explains why the algebraicity of p-adic singular moduli can be approached unconditionally.

This crucial ingredient is *entirely lacking* in the archimedean setting. This explains why the p-adic Gross-Stark conjectures are more tractable than their archimedean counterparts.

An analogous formula for GL_n

Let T_1 and T_2 be maximal tori in GL_n, for which

$$T_1(\mathbb{Q}) = K_1^\times, \qquad T_2(\mathbb{Q}) = K_2^\times,$$

with K_1 and K_2 totally real fields of degree n in which p is *inert*.

Let $v_1 \in K_1^n$ and $v_2 \in K_2^n$ be eigenvectors for $T_1(\mathbb{Q})$ and $T_2(\mathbb{Q})$ respectively.

Goal: to define a p-adic quantity $J_p(v_1, v_2)$ generalising the previous one for $n = 2$.

Archimedean cycles

Archimedean symmetric space of dimension $(n^2 + n - 2)/2$:

$X_\infty = \mathrm{SL}_n(\mathbb{R})/SO_n(\mathbb{R}) = \{$ positive definite quadratic forms on $\mathbb{R}^n\}$.

After choosing real embeddings of K_1 and K_2, we obtain vectors

$$v_1^{(1)}, \ldots, v_1^{(n)} \in \mathbb{R}^n, \qquad v_2^{(1)}, \ldots, v_2^{(n)} \in \mathbb{R}^n.$$

Topological cycles attached to T_1 and T_2:

- $\delta(T_1) := \{q \in X_\infty \text{ s.t. } q(v_1^{(i)}, v_1^{(j)}) = 0, \ \forall\, i \neq j.\}$. It is a cycle of real dimension $n-1$ in X_∞.

- $\Delta(T_2) = \{q \in X_\infty \text{ s.t. } q(v_2^{(1)}, v_2^{(j)}) = 0, \ \forall\, 2 \leq j \leq n.\}$ It is a cycle of codimension $n-1$ in X_∞.

Topological intersection on X_∞: $(\delta(T_1) \cdot \Delta(T_2)) \in \{-1, 0, 1\}$.

A local Green's function

Let $\quad v_1, v_1^{(2)}, \ldots, v_1^{(n)} \in \mathbb{C}_p^n, \qquad v_2, v_2^{(2)}, \ldots, v_2^{(n)} \in \mathbb{C}_p^n$

be the *frobenius translates* of v_1 and v_2, viewed as vectors in $\mathbb{C}_p^n$.

The *local Green's function* $g(v_1, v_2)$ attached to v_1 and v_2 is

$$\left\{ \frac{\det(v_1; v_2^{(2)}; \ldots; v_2^{(n)}) \cdot \det(v_2; v_1^{(2)}; \ldots; v_1^{(n)})}{\det(v_1; v_1^{(2)}; \ldots, v_1^{(n)}) \cdot \det(v_2; v_2^{(2)}; \ldots, v_2^{(n)})} \right\}^{(\delta(T_1) \cdot \Delta(T_2))} \in K_1 K_2.$$

$$g(v_1, v_2) = (v_2^*(v_1) \cdot v_1^*(v_2))^{(\delta(T_1) \cdot \Delta(T_2))}.$$

This function is a point-pair invariant:

$$g(\alpha v_1, \alpha v_2) = g(v_1, v_2), \qquad \text{for all } \alpha \in \mathrm{SL}_n(\mathbb{Q}).$$

The global Green's function

The group $\Gamma := \mathsf{SL}_n(\mathbb{Z})$ acts on the tori by conjugation, and on $M_n(\mathbb{Z})$ by left and right multiplication.

$$\Gamma_1 := \mathrm{Stab}_\Gamma(v_1) = T_1(\mathbb{Z}), \qquad \Gamma_2 := \mathrm{Stab}_\Gamma(v_2) = T_2(\mathbb{Z}).$$

$$G_m(\tau_1, \tau_2) = \prod_{\substack{\alpha \in \Gamma_1 \backslash M_n(\mathbb{Z}) / \Gamma_2 \\ \det(\alpha) = m}} g(v_1, \alpha v_2) \quad \in \quad K_1 K_2.$$

This quantity is a finite product of local Green's functions, and belongs to the field $K_1 K_2$.

$$G_m^{(n)}(v_1, v_2) = G_{mp^{2n}}(v_1, v_2).$$

p-adic limits

Choose a p-adic embedding $K_1 K_2 \hookrightarrow \mathbb{Q}_{p^n}$.

$$g(v_1, v_2), \quad G_m^{(n)}(v_1, v_2) \in \mathbb{Q}_{p^n}.$$

Given a formal combination $\eta = \sum a_j [m_j]$, let

$$G_\eta^{(n)}(v_1, v_2) := \prod_j G_{m_j}^{(n)}(v_1, v_2)^{a_j};$$

p-adic class invariants for $\mathrm{GL}(n)$?

Conjecture (Charollois, D)

There exists $\eta = \sum a_j [m_j]$ *for which the limit*

$$J_\eta(v_1, v_2) := \lim_{n\to\infty} G_\eta^{(n)}(v_1, v_2)$$

exists in $\mathbb{Q}_{p^n}$*, for any tori* T_1 *and* T_2*.*
It belongs to $H_1 H_2$*, up to torsion, where* H_j *is the narrow class field of* K_j*.*

This conjecture is very speculative, lacking in theoretical or experimental evidence, and barely deserves to be called a conjecture at all.

What happens in Nisyros, stays in Nisyros!

Conclusion

The traditional approach to explicit class field theory via CM theory is based on the CM values of modular functions: classes in $H^0(\Gamma, \mathcal{A}(X))$ where Γ is an arithmetic group and $X = G(\mathbb{R})/K$ is a symmetric space.

One begins to discern the bare outline of a more general framework, in which modular functions are replaced by (analytic, meromorphic, rigid meromorphic...) cocycles.

Until BCG, the picture was confined largely to GL_2. A more general theory for GL_n is gradually beginning to emerge.

It remains largely speculative, and a lot remains to be understood...

Thank you for your attention!