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Motives and automorphic representations

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Beyond Endoscopy: (1) A strategy proposed by Langlands (2000) to use the stable trace formula to prove Langlands' Functoriality conjecture, and the analytic continuation and functional equation of all automorphic L-f^ns $L(s, \pi, r)$

(2) Further hope is that, combined with generalizations of techniques of Taylor-Wiles, it will lead eventually to a proof of Langlands Reciprocity conjecture — a general analogue of the STW conjecture, and the analytic cont. + f^nal eq^tn of all motivic L-f^ns $L(s, M)$

Motives (pure) Grothendieck (c 1965) have two simultaneous roles.

(i) Fundamental (but hidden) building blocks of smooth projective varieties

(ii) Universal cohomology theory, through which all other cohom. theories ~~in~~ alg. geom. factor (Betti, de Rham, l-adic, Hodge etc.)

F, Q fields of char. 0, with $F \subset \mathbb{C}$. (F usually a number field + Q is usually $\mathbb{Q}$). Then we expect to have a category $\mathrm{Mot}_{F,Q}$, the semi-simple, Q-linear category of pure motives over F with coeff^s in Q. It would come with two functors

$$(S\,\mathrm{Proj})_F \xrightarrow{M_F} \mathrm{Mot}_{F,Q}$$

smooth, proj varieties over F

and

$$\mathrm{Mot}_{F,Q} \xrightarrow{R_B} (\mathrm{Vect})_Q,$$

whose composition

$$H_B = R_B \circ M_F,$$

$$(S\,\mathrm{Proj})_F \longrightarrow \mathrm{Mot}_{F,Q} \longrightarrow (\mathrm{Vect})_Q,$$

building blocks   universal cohomology

is just Betti (singular) cohom. of complex varieties with Q-coeff^s; $\mathrm{Mot}_{F,Q}$ would be a Tannakian category with fibre functor R_B. This means that $\mathrm{Mot}_{F,Q}$ is ~~(anti)~~ ~~iso^c~~ du… to the category $\mathrm{Rep}_Q(G_{F,Q})$ of finite dim rep^s of a reductive, proalg. gp defined over Q.

Assume F is a number field + $Q = \mathbb{Q}$. Then $G_F = G_{F,\mathbb{Q}}$ is a gp over $\mathbb{Q}$, with canonical mapping $G_F \to \Gamma_F$, for the Galois gp $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$.

What is it?

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Automorphic Representations (Langlands (1967)) should be governed by certain locally compact gp^s attached to F & its local $comp^{ns}$ F_v

$$\begin{tikzcd} L_{F_v} \arrow[r, two heads] \arrow[d, hook] & W_{F_v} \arrow[r] \arrow[d, hook] & \Gamma_{F_v} \arrow[d, hook] \\ L_F \arrow[r, two heads] & W_F \arrow[r] & \Gamma_F \end{tikzcd}$$

embeddings defined up to conjugacy

Langlands gp^s — Weil gp^s — Galois gp^s ,

for the local Langlands group $L_{F_v} = \begin{cases} W_{F_v}, & \text{if } v \text{ archimedean} \\ W_{F_v} \times SU(2), & v \text{ discrete,} \end{cases}$

& the hypothetical global Langlands group L_F (aut. Gal. gp)

Candidates for L_F: 2 ingredients

(i) Indexing set $\mathcal{C}_F$: Recall that

$\pi \in \Pi_{aut}(G)$ (an automorphic rep. of reductive gp G/F)

$\rightsquigarrow$ family $c(\pi) = \{ c_v(\pi) = c(\pi_v) : v \notin S \}$ (local $comp^{t}$ of π) (finite set of places)

of s.s. $\hat{G}_v$ - orbits in (dual gp of G_v)

${}^L G_v = \hat{G}_v \rtimes \Gamma_{F_v}$ (L-gp of G_v). Let $\mathcal{C}_{aut}(G) = \{ c(\pi) : \pi \in \Pi_{aut}(G) \} / \sim$

~~be the set of such families c~~ (where $c' \sim c$ if $c'_v = c_v$ for a.a. v)

If G/F is quasi-split, simple + simply conn, let

$\mathcal{C}_{prim}(G)$ be the set of $c \in \mathcal{C}_{aut}(G)$ $\ni$

$\mathrm{ord}_{s=1}(L^S(s, c, r)) = [r : 1_G]$ (mult. of triv. rep in r)

$\forall\ r : {}^L G \longrightarrow GL(N, \mathbb{C})$. Then define $\mathcal{C}_F$ to be the

set of iso^{m} class of pairs $\{ (G, c) : c \in \mathcal{C}_{prim}(G) \}$.

(ii) For any $c \in \mathcal{C}_F$, an ext^n $1 \to K_c \to L_c \to W_F \to 1$ of W_F by a compact s.c. gp K_c: Given $c \sim (G, c)$ in $\mathcal{C}_F$, let K_c be the compact real form of the s.c. cover $\hat{G}_{sc}$ of the (adjoint) dual gp $\hat{G}$ of G. ~~From this~~, not hard to construct required ext^n

$$1 \to K_c \to L_c \to W_F \to 1$$

Note: Both (i) & (ii) depend on functorialities (for the values of $L^S(s, c, r)$ near $s=1$) and (for) a little more (which ought to become clear with a successful proof of functoriality by BE

Def: Autonomorphic Galois gp

$$L_F = \prod_{c \in \mathcal{C}_F} (L_c \to W_F)$$ - fibre product over W_F

- a loc cp^t gp, with local embeddings

$$\begin{array}{ccc} L_{F_v} & \longrightarrow & W_{F_v} \\ \downarrow & & \downarrow \\ L_F & \longrightarrow & W_F \end{array}$$

Conjecture 1: $\exists$ a bijection

$\{$ Irred rep^s $r: L_F \to GL(N, \mathbb{C})\} \to \{$cusp aut rep^s of $GL(N, \mathbb{A})\}$,

compatible with the localizations F_v

Conjecture 2: A more general mapping

$$\{\phi: L_F \to {}^L G\} \longrightarrow \{\Pi_\phi \subset \Pi_{aut, temp}(G)\}, \quad G/F \text{ q.s.},$$

from b^{dd}, global Langlands parameters to disjoint, global L-packets, whose union is $\Pi_{aut, temp}(G)$

(the set of tempered, aut rep^s of G.)

5

A conjectural candidate for motivic Galois gp $\mathcal{G}_F$

The const^{n} of L_F leads to a complex, pro-algebraic group $\mathcal{G}_F$, depending on the embedding $F \subset \mathbb{C}$, with maps

$$
\begin{tikzcd}
L_F \arrow[r] \arrow[d] & W_F \arrow[r] \arrow[d] & T_F \arrow[d] & \text{(loc. cpt gps)} \arrow[l] \\
\mathcal{G}_F \arrow[r] & \mathcal{T}_F \arrow[r] & \Gamma_F & \text{(proalg. gps)} \arrow[l]
\end{tikzcd}
$$

, whose rep^{s} $r : \mathcal{G}_F \to GL(n, \mathbb{C})$ should parametrize motives.

Fibre product of complex^{n} of those L_C of Hodge type — i.e. $\forall\, r : L_C \to GL(n, \mathbb{C})$ & $\forall\, v \in S_\infty$, the rest^{n} of r to the sub^{gp} $\mathbb{C}^* \subset W_v \hookrightarrow L_C$ is a Hodge structure

Langlands Taniyama gp — Corvallis — "algebraic hull of the motivic part of W_F"

This would be a huge generalization of the STW conj. for elliptic curves over $\mathbb{Q}$, whose proof we can hope is related to BE.

Mere work needed on the conjectural const^{n} here

(i) The motivic Galois gp $\mathcal{G}_F$ is supposed to be defined over $\mathbb{Q}$, while the group we have defined here is complex. Figure out its structure over $\mathbb{Q}$ from the families (G, c) in $\mathcal{C}_F$ that parametrize the factors of L_F & hence of $\mathcal{G}_F$

(ii) Try to figure out the cohomology realizations of $\mathcal{G}_F$, again from the families (G, c) in $\mathcal{C}_F$.

<u>This is Just a Beginning</u>: We should also have the <u>mixed motivic Galois gp</u> $\mathcal{G}_F^+$, with

$$\mathcal{G}_F \ltimes \mathcal{U}_F = \mathcal{G}_F^+ \to \mathcal{G}_F \to \mathcal{J}_F \to \Gamma_F,$$

which plays the role of $\mathcal{G}_F$ for sing./open varieties.

<u>Problem</u>: Find an explicit (conjectural) formula for its unip. radical $\mathcal{U}_F$, as a prounip. gp with an action of $\mathcal{G}_F$. — closely related to the Beilinson conjectures. Would we also like a (locally compact?) automorphic analogue $L_F \ltimes U_F$, constructing U_F in terms of Eisenstein series?

· We also have the exponential motivic Galois gp $\widehat{\mathcal{G}}_F$, + its mixed extension $\widehat{\mathcal{G}}_F^+$, with

ext^n of $\mathcal{J}_F$ defined by Greg Anderson

$$\begin{array}{ccccccc}
\widehat{\mathcal{G}}_F \ltimes \widehat{\mathcal{U}}_F = \widehat{\mathcal{G}}_F^+ & \longrightarrow & \widetilde{\mathcal{G}}_F & \longrightarrow & \widehat{\mathcal{J}}_F & \longrightarrow & \Gamma_F \\
\downarrow \qquad\qquad \downarrow & & \downarrow & & \downarrow & & \| \\
\mathcal{G}_F \ltimes \mathcal{U}_F = \mathcal{G}_F^+ & \longrightarrow & \mathcal{G}_F & \longrightarrow & \mathcal{J}_F & \longrightarrow & \Gamma_F
\end{array}$$

<u>Problem</u>: Find an explicit (automorphic) expression for $\widehat{\mathcal{G}}_F$ like what we have for $\mathcal{G}_F$. For this, we would seem to need an exponential aut. Galois gp $\widehat{L}_F \to L_F$. An enormous problem, but

<u>Conjecture</u>: $\widehat{L}_F$ ~~should~~ comes from the aut. rep^s of covering groups $\widehat{G(\mathbb{A})}$, obtained by Brylinski – Deligne extensions.