The fine spectral expansion of the Rankin–Selberg period
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1 Context

1.1 The Rankin–Selberg period

Let F be a number field. Set $G = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$ and $H = \mathrm{GL}_n$ embedded diagonally into G. Write $[H] = H(F)\backslash H(\mathbb{A})$.

For π a cuspidal representation of G and $\varphi \in \pi$, the *Rankin–Selberg* period is

$$\mathcal{P}_H(\varphi) = \int_{[H]} \varphi(h)dh.$$

We have the following result.

Theorem 1.1.1 (Jacquet, Piatetski-Shapiro, Shalika). *If $\varphi = \otimes_v \varphi_v \in \pi$ cuspidal, then*

$$\mathcal{P}_H(\varphi) \sim L(1/2, \pi) \prod_{v \in S} \mathcal{P}_v(\varphi_v), \tag{1.1.1}$$

where S is a finite set of places of F, and the equality holds for explicit choices of measures.

1.2 The Jacquet–Rallis relative trace formula

Let E/F be quadratic. Write $G_E = \mathrm{Res}_{E/F}(\mathrm{GL}_{n,E} \times \mathrm{GL}_{n+1,E})$ and $H_E = \mathrm{Res}_{E/F}\mathrm{GL}_{n,E}$. The Jacquet–Rallis trace formula is taken relatively to $H_E\backslash G_E/G$. More precisely, for $f \in C_c^\infty(G_E(\mathbb{A}))$ its spectral expansion is (if we ignore all convergence issues)

$$\int_{[H_E]} \int_{[G]} K_f(h,g)\eta(g)dgdh = \sum_{\pi_E \text{ cusp. of } G_E} \sum_{\varphi \in \mathrm{ONB}(\pi_E)} \mathcal{P}_{H_E}(\pi_E(f)\varphi)\overline{\mathcal{P}_{G,\eta}(\varphi)} + (\dots), \tag{1.2.1}$$

where η is some quadratic character and $\mathcal{P}_{G,\eta}$ is the *Flicker–Rallis* period which detects the image of base change from unitary groups. The $(\dots)$ corresponds to the non-cuspidal terms.

Jacquet and Rallis proposed to relate this relative trace formula to another one corresponding to $\mathrm{U}(V_n)\backslash \mathrm{U}(V_n) \times \mathrm{U}(V_{n+1})/\mathrm{U}(V_n)$, where V_n and V_{n+1} are Hermitian spaces of dimensions n and $n+1$. The spectral expansion of this trace formula makes the period along $[\mathrm{U}(V_n)]$ appear. According to the Gan–Gross–Prasad conjecture, if π is a cuspidal representation of $\mathrm{U}(V_n) \times \mathrm{U}(V_{n+1})$ we should have a relation

$$\left(\int_{[\mathrm{U}(V_n)]} \varphi(h)dh, \quad \varphi \in \pi\right) \longleftrightarrow L(1/2, \mathrm{BC}(\pi)).$$

By combining the equality (1.1.1) given by the theorem of JPSS and the spectral expansion (1.2.1) of the RTF on the GL-side, we can hope to prove the conjecture by comparing the two RTFs.

Question: How much do we need to know about the non-cuspidal part $(\dots)$ in (1.2.1) ?

If π is cuspidal, $\mathrm{BC}(\pi)$ might not be, so it is necessary to go beyond a simple trace formula. In 2022, Beuzart-Plessis, Chaudouard and Zydor proved the GGP conjecture for the π's such that $\mathrm{BC}(\pi)$ is generic. In particular, they computed the generic part of the $(\dots)$ terms of (1.2.1).

1.3 This talk

Question: What about the non-generic terms in $(\dots)$?

In the framework of GGP, understanding these non-generic terms is related to the non-generic conjecture formulated by Gan, Gross and Prasad in 2020. In this talk, we will give some results towards the computation of the fine spectral expansion of the Jacquet–Rallis trace formula (1.2.1). More precisely, we focus on the integral associated to the Rankin–Selberg period, so that our goal is to compute the spectral expansion of the distribution

$$f \in \mathcal{S}([G_E]) \mapsto \int_{[H_E]} f(h)dh. \tag{1.3.1}$$

We will do so in two parts:

1. We will describe periods associated to non-generic representations that will appear in the expansion of (1.3.1). This extends the results of JPSS.

2. We will write the fine spectral expansion of (1.3.1).

For the Flicker–Rallis period $\mathcal{P}_{G,\eta}$, this work has been carried out by Chaudouard in a recent paper. In a future work we hope to combine our results with his to compute the expansion of the trace formula itself.

2 Non-generic Rankin–Selberg periods

We now forget about the extension E/F, so that we go back to $G = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$ and $H = \mathrm{GL}_n$.

2.1 Relevant representations of Arthur type

We want to extend $\mathcal{P}_H$ to non-generic representations. We are interested in the ones appearing in the trace formula, i.e. in the constituents (either discrete or continuous) of $L^2([G])$. We say that these automorphic representations are of *Arthur type*, and we denote their set of isomorphism classes by $\Pi_{\mathrm{Art}}(G)$.

If it exists, the extension of $\mathcal{P}_H$ to a $\pi \in \Pi_{\mathrm{Art}}(G)$ might be zero for local reasons as there may not exist a non-zero $H(F_v)$-invariant linear form on π_v. If such a linear form exists, we say that π_v is *distinguished*. Gan Gross and Prasad have given a conjectural characterization of these representations in terms of local Arthur parameters (which are said to be *relevant*). It is now a theorem of Gurevitch and Chan at non-Archimedean places, and Chen and Chen at Archimedean places (in fact they use Theorem 2.3.1 below). In any case, by globalizing this local relevance condition we obtain a global one. We denote the set of *relevant* automorphic representations of Arthur type by $\Pi^H_{\mathrm{Art}}(G)$. We now describe it.

If σ is a cuspidal representation of some GL_r, we define the derivative of $\mathrm{Speh}(\sigma, d)$ by

$$\mathrm{Speh}(\sigma, d)^- = \mathrm{Speh}(\sigma, d-1).$$

If $d = 1$, then $\mathrm{Speh}(\sigma, d)^-$ is the trivial representation of the trivial group. We then declare that $\pi = \pi_n \boxtimes \pi_{n+1} \in \Pi^H_{\mathrm{Art}}(G)$ if π is a parabolic induction of the form

$$\pi_n = (\pi_{1,1} \times \ldots \times \pi_{1,m_1}) \times \left(\pi_{2,1}^{-,\vee} \times \ldots \times \pi_{2,m_2}^{-,\vee}\right), \tag{2.1.1}$$

$$\pi_{n+1} = \left(\pi_{1,1}^{-,\vee} \times \ldots \times \pi_{1,m_1}^{-,\vee}\right) \times (\pi_{2,1} \times \ldots \times \pi_{2,m_2}), \tag{2.1.2}$$

where the $\pi_{1,i}$ and $\pi_{2,i}$ are all Speh representations.

We have the following important examples.

- Cuspidal representations π are relevant. The above decomposition reduces to $\pi_n = \pi_{1,1}$ and $\pi_{n+1} = \pi_{2,1}$ cuspidal.
- Generic representations π in $\Pi_{\mathrm{Art}}(G)$ are relevant. Indeed, by the classification of the discrete spectrum from Mœglin and Waldspurger, this is equivalent to $\pi_n = \pi_{1,1} \times \ldots \times \pi_{1,m_1}$ and $\pi_{n+1} = \pi_{2,1} \times \ldots \times \pi_{2,m_2}$, where the $\pi_{1,i}$ and $\pi_{2,i}$ are all cuspidal.

2.2 A special value of L-functions

Let $\pi \in \Pi^H_{\mathrm{Art}}(G)$. For $a \in \{1, 2\}$, write $\pi_{a,i} = \mathrm{Speh}(\sigma_{a,i}, d_{a,i})$. We then define

$$\mathcal{L}(\pi) = \frac{\prod_{i,j} L^*\left(\frac{d_{1,i}-d_{2,j}+1}{2}, \sigma_{1,i} \times \sigma_{2,j}\right)}{\prod_{a\in\{1,2\}} \prod_{i<j} L\left(\frac{d_{a,i}+d_{a,j}}{2}, \sigma_{a,i} \times \sigma_{a,j}^\vee\right)},$$

where L^* is the residue of the L-function if it has a pole at that point. Note that the notation is slightly abusive as $\mathcal{L}(\pi)$ depends on the inducing datum and not only on the isomorphism class of π. We have two examples.

- If π is cuspidal, then $\mathcal{L}(\pi) = L(1/2, \pi)$.
- If $\pi = I_P^G \sigma$ with σ cuspidal, then $\mathcal{L}(\pi)$ is the quotient of L-functions appearing in the Euler product of Rankin–Selberg Zeta integrals induced from σ, i.e. for $\varphi \in \otimes_v \varphi_v \in \pi$

  $$Z(\varphi, \lambda) = \mathcal{L}(I_P^G \sigma_\lambda) \prod_{v\in S} Z_v^\sharp(\varphi_v, \lambda),$$

  where the $Z_v^\sharp(\varphi_v, \lambda)$ are normalized local Zeta integrals.

2.3 The theorem

The following result is the non-generic GGP conjecture for Rankin–Selberg periods (i.e. the split Bessel case).

Theorem 2.3.1. *The following assertions hold.*

1. *The period $\mathcal{P}_H$ extends naturally to $\Pi_{\mathrm{Art}}^H(G)$.*
2. *For $\pi \in \Pi_{\mathrm{Art}}^H(G)$ and $\varphi = \otimes_v \varphi_v \in \pi$, we have*

$$\mathcal{P}_H(\varphi) \sim \mathcal{L}(\pi) \prod_{v \in S} \mathcal{P}_v(\varphi_v).$$

3. *The local periods $\mathcal{P}_v$ are non-zero on π_v.*

2.4 Construction of $\mathcal{P}_H$

In 2015, Ichino and Yamana built an extension $\mathcal{P}_H^{IY}$ of $\mathcal{P}_H$, defined (almost) on $\Pi_{\mathrm{Art}}(G)$. Moreover, if $\pi = I_P^G \sigma \in \Pi_{\mathrm{Art}}(G)$ with σ discrete (non-necessarily cuspidal), they showed that it computes Zeta functions, i.e. for $\varphi \in \pi$

$$\mathcal{P}_H^{IY}(E(\varphi, \lambda)) = Z(\varphi, \lambda). \tag{2.4.1}$$

However, because $Z(\varphi, \lambda)$ is defined in terms of a Whittaker function, this implies that $\mathcal{P}_H^{IY}(E(\varphi, \lambda))$ is zero as soon as π is not generic.

To circumvent this issue, we use the following strategy. Let $\pi \in \Pi_{\mathrm{Art}}^H(G)$. We know that it is a quotient of some $I_P^G \sigma_\nu$ for σ cuspidal and $\nu \in \mathfrak{a}_P^*$ by taking residues of Eisenstein series. We want to prove a factorization of the form

$$\begin{array}{ccc} I_P^G \sigma_\nu & \xrightarrow{E^*} & \pi \\ {\scriptstyle \operatorname{Res}_{\lambda=\nu} \mathcal{P}_H^{IY}(E(\varphi,\lambda))} \downarrow & \swarrow {\scriptstyle \mathcal{P}_H} & \\ \mathbb{C} & & \end{array} \tag{2.4.2}$$

We now explain which residues to consider.

The natural choice is to take residues along the singularities of the Eisenstein series E at ν. However, we can show that along such singularities we have

$$\operatorname*{Res}_{\lambda=\nu} \mathcal{P}_H^{IY}(E(\varphi, \lambda)) = \mathcal{P}_H^{IY}(\operatorname*{Res}_{\lambda=\nu} E(\varphi, \lambda)) = 0, \tag{2.4.3}$$

where the last equality comes from the fact that residual Eisenstein series are non-generic. So this is not the correct method.

It turns out that the meromorphic map $\lambda \mapsto \mathcal{P}_H^{IY}(E(\varphi, \lambda))$ has other residues passing through ν. Indeed, $\mathcal{P}_H^{IY}$ is defined à la Jacquet–Lapid–Rogawski as the pure polynomial term of some truncated period $T \mapsto \mathcal{P}_H^T$. This pure polynomial term will only be constant if λ lies outside of some hyperplanes (what really matters here are the cuspidal exponents of $E(\varphi, \lambda)$). These hyperplanes yield singularities of $\lambda \mapsto \mathcal{P}_H^{IY}(E(\varphi, \lambda))$. By taking the residues along these hyperplanes, we can show that

$$\operatorname*{Res}_{\lambda=\nu} \mathcal{P}_H^{IY}(E(\varphi, \lambda)) = \mathcal{P}_H^Q E^Q(M^*(w, \nu)\varphi, w\nu). \tag{2.4.4}$$

Here $\mathcal{P}_H^Q$ is some *relative* regularized period which is a generalization of $\mathcal{P}_H^{IY}$ relatively to a parabolic subgroup $Q \subset G$, and $E^Q(M^*(w,\nu)\varphi, w\nu)$ is a partial Eisenstein series applied to a regularized global intertwining operator $M^*(w,\nu)\varphi$, and w some element in the Weyl group. To prove (2.4.4), one uses the fact that regularized and truncated periods can be related to one another using *Maass–Selberg* relations, which makes the computation of the residue possible.

The slogan is that **the residue of a regularized period $\mathcal{P}_H^{IY}(E(\varphi,\lambda))$ is a relative regularized period $\mathcal{P}^Q$ applied to parts of the constant term of $E(\varphi,\lambda)$ along Q.**

It turns out that $M^*(w,\nu)$ also realizes the quotient $I_P^G\sigma_\nu \to \pi$ in (2.4.2), so that we indeed obtain the desired factorization. Therefore, $\operatorname{Res}_{\lambda=\nu} \mathcal{P}_H^{IY}(E(\varphi,\lambda))$ defines the extension $\mathcal{P}_H$ of the first point of Theorem 2.3.1.

We highlight the differences between the two kind of residues in an example. For $G = \mathrm{GL}_1 \times \mathrm{GL}_2$, consider the induction

$$I_\lambda = |\cdot|^{\lambda_1} \boxtimes \left(|\cdot|^{\lambda_2} \times |\cdot|^{\lambda_3}\right).$$

Using the factorization of Zeta functions, for $\varphi \in I_0$ we have

$$\mathcal{P}_H^{IY}(E(\varphi,\lambda)) = \frac{\zeta_F(\lambda_1+\lambda_2+1/2)\zeta_F(\lambda_1+\lambda_3+1/2)}{\zeta_F(\lambda_2-\lambda_3+1)} \prod_{v\in S} Z_v^\sharp(\varphi,\lambda).$$

There is no singularity along $\lambda_2 - \lambda_3 = 1$, so that we recover (2.4.3). The residues taken in (2.4.4) are along $\lambda_1 + \lambda_2 = 1/2$ and $\lambda_1 + \lambda_3 = -1/2$.

To prove the second point of Theorem 2.3.1, i.e. the Euler product expression, we use Zeta functions. Indeed, by the main theorem of Ichino and Yamana we have

$$\operatorname*{Res}_{\lambda=\nu} \mathcal{P}_H^{IY}(E(\varphi,\lambda)) = \operatorname*{Res}_{\lambda=\nu} Z(\varphi,\lambda) = \left(\operatorname*{Res}_{\lambda=\nu} \mathcal{L}(I_P^G\sigma_\lambda)\right) \prod_{v\in S} Z_v^\sharp(\varphi,\lambda) = \mathcal{L}(\pi) \prod_{v\in S} Z_v^\sharp(\varphi,\lambda). \tag{2.4.5}$$

This gives the Euler product expansion of Theorem 2.3.1. Note that the two distinct computations of the residue yield different information about $\mathcal{P}_H$, and are both necessary in the proof.

2.5 About local Zeta functionals

The expression in (2.4.5) tells us what the linear forms $\mathcal{P}_v$ of Theorem 2.3.1 are: they are normalized local Zeta integrals, a priori defined on the induction $I_P^G\sigma_{v,\nu}$, and it turns out that they factorize through the quotient $I_P^G\sigma_{v,\nu} \to \pi_v$ as they do so globally (see (2.4.4)).

It remains to prove that they are non-zero. This follows from this factorization property combined with a result of JPSS. This proves the "relevance implies distinction" part of the local non-generic GGP for split Bessel periods (this result was already known by different methods in the p-adic case, but is new in the Archimedean case). Note that here we a global argument to prove that $Z_v^\sharp$ passes through the quotient $I_P^G\sigma_{v,\nu} \to \pi_v$, but an alternative proof using purely local arguments can be given in the p-adic case using asymptotics of Whittaker functions.

3 The fine spectral expansion of the Rankin–Selberg period

3.1 Enhanced relevant representations of Arthur type

We now go back to our initial goal, which was to compute the spectral expansion of

$$f \in \mathcal{S}([G]) \mapsto \int_{[H]} f(h)dh.$$

We could expect that it involves the relevant representations in $\Pi^H_{\mathrm{Art}}(G)$, but surprisingly other representations appear. To describe them, we introduce the set $\Pi^H_{\mathrm{Art}}(G)^+$ of *enhanced relevant representations of Arthur type*.

More precisely, we say that $\pi = \pi_n \boxtimes \pi_{n+1}$ belongs to $\Pi^H_{\mathrm{Art}}(G)^+$ if it can be writen as an induction

$$\pi_n = (\pi_{+,1} \times \ldots \times \pi_{+,m_+}) \times \pi_k \times (\pi_{-,1} \times \ldots \times \pi_{-,m_-}),$$
$$\pi_{n+1} = \left(\pi^\vee_{+,1} \times \ldots \times \pi^\vee_{+,m_+}\right) \times \pi_{k+1} \times \left(\pi^\vee_{-,1} \times \ldots \times \pi^\vee_{-,m_-}\right),$$

where $\pi_k \boxtimes \pi_{k+1}$ is a relevant representation of Arthur type in $\Pi^H_{\mathrm{Art}}(\mathrm{GL}_k \times \mathrm{GL}_{k+1})$, and the $\pi_{+,i}$ and $\pi_{-,i}$ are Speh. We write $\pi = I_P^G \tau$, where τ is a product of Speh's. We also impose that all the central characters of the $\pi_{.,i}$ are trivial on the Archimedean components of the split centers of each GL. Then if we write $\pi_k \boxtimes \pi_{k+1}$ as in (2.1.1) and (2.1.2), we can write the Levi M_P as

$$M_P = \left(M_+ \times M_1 \times M_2^- \times M_-\right) \times \left(M_+ \times M_1^- \times M_2 \times M_-\right),$$

where $\boxtimes_{i=1}^{m_+} \pi_{+,i}$ is a representation of M_+, $\boxtimes_{i=1}^{m_1} \pi_{1,i}$ of M_1 and so on. Relatively to this decomposition we define a subspace $\mathfrak{a}^*_\pi$ of unramified characters of P by

$$\mathfrak{a}^*_{P,\mathbb{C}} \supset \mathfrak{a}^*_{\pi,\mathbb{C}} = \left\{((\lambda_+, \lambda_1, -\lambda_2, \lambda_-), (-\lambda_+, -\lambda_1, \lambda_2, -\lambda_-))\right\}.$$

Then $i\mathfrak{a}^*_\pi$ is the subspace of unitary unramified characters such that $\pi_\lambda := I_P^G \tau_\lambda$ is still relevant (in the enhanced sense) if we lift the condition on the central characters. Finally, we define a non-unitary unramified character ρ_π by the formula

$$\rho_\pi = ((1/4, 0, 0, -1/4), (1/4, 0, 0, -1/4)) \in \mathfrak{a}^*_P, \tag{3.1.1}$$

where we still use the same coordinates. Note that if $\pi \in \Pi^H_{\mathrm{Art}}(G)$, so that $m_+ = m_- = 0$, ρ_π is zero.

We claim that $\mathcal{P}_H$ can be extended to π_λ for $\pi \in \Pi^H_{\mathrm{Art}}(G)^+$ and $\lambda \in \mathfrak{a}^*_{\pi,\mathbb{C}} - \rho_\pi$. More precisely, for $\varphi \in \pi$, the map $\lambda \mapsto \mathcal{P}_H(\varphi, \lambda)$ is a meromorphic function on $\mathfrak{a}^*_{\pi,\mathbb{C}} - \rho_\pi$. The twist by ρ_π is necessary here, as otherwise the induction might not have a non-zero $H(\mathbb{A})$-invariant linear form.

3.2 The result

Our result is the following. It is a work in progress at the proof is not released as of today.

Theorem 3.2.1 (WIP)**.** *For $f \in \mathcal{S}([G])$, we have*

$$\int_{[H]} f(h)dh = \sum_{\pi \in \Pi^H_{\mathrm{Art}}(G)^+} \sum_{\varphi \in \mathrm{ONB}(\pi)} c_\pi \int_{i\mathfrak{a}^*_\pi - \rho_\pi} \langle f, E(\varphi, -\overline{\lambda})\rangle \mathcal{P}_H(\varphi, \lambda) d\lambda,$$

where the c_π are some explicit constants and

$$\langle f, E(\varphi, -\overline{\lambda})\rangle = \int_{[G]} f(g)\overline{E(g, \varphi, -\overline{\lambda})} dg. \tag{3.2.1}$$

Here are some remarks on the result.

- The choice of ρ_π is somehow arbitrary, as it is really the space $\mathfrak{a}^*_{\pi,\mathbb{C}} - \rho_\pi$ that matters here (which means that the sums of the first and last coordinates in (3.1.1) have to be $1/2$, and $-1/2$). However, in the formula of (3.2.1) we also want to avoid possible poles of Eisenstein series, so that we have to restrict to ρ_π of the form

$$\rho_\pi = ((t, 0, 0, -s), (1/2 - t, 0, 0, -1/2 + s)), \quad 0 < t, s < 1/2.$$

- The representations π_λ which support the expansion (3.2.1) might not be unitary.

- The function $\lambda \mapsto \mathcal{P}_H(\varphi, \lambda)$ may have poles in the region of integration, but the product $\langle f, E(\varphi, -\overline{\lambda})\rangle \mathcal{P}_H(\varphi, \lambda)$ is regular.

3.3 About the proof

The general idea is to proceed by shift of contours, following the philosophy of the proof of the spectral decomposition of the scalar product by Langlands, starting from pseudo-Eisenstein series. The main ingredients are of the following nature.

- For the convergence of the integrand (and its holomorphicity), we use the existence of zero-free regions of Rankin–Selberg L-functions (due to Brumley and Lapid) and of bounds towards the generalized Ramanujan conjecture (Luo, Rudnick, Sarnak).

- During the shift of contours, we will catch some poles. They may come either from the regularized period $\mathcal{P}_H$, or from the Eisenstein series $E(\varphi, -\overline{\lambda})$. As we explained, the period $\mathcal{P}_H$ is built via residues, so that the residues of $\mathcal{P}_H$ appearing in the proof of the theorem will be new regularized periods. These contributions are not too difficult to understand. In contrast, we need some precise results on the singularities of the residual Eisenstein series $E(\varphi, -\overline{\lambda})$. It turns out that they are more or less contained in the paper of Mœglin and Waldspurger on the discrete spectrum of GL_n.

- The shift of contours can be realized in such a way that a some point a spectral decomposition of a scalar product appears. It is then possible to use the theorem of Langlands as an input, which avoids some very difficult computation. In particular, by proceeding this way it turns out that all the residues that are obtained through the shift contribute in the final formula (3.2.1), so that in some sense the intricate compensations are taken care of by Langlands' result.