Local Langlands correspondence and affine Hecke algebras

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A pictorial description I

The situation you will consider:

Two collections of objects of different nature, that are conjecturally related via the Local Langlands Correspondence:

$$\left\{\begin{matrix}\text{irred. smooth repns.}\\ \text{of } p\text{-adic groups}\end{matrix}\right\}/\text{equiv.} \xleftrightarrow{\text{LLC}} \left\{\begin{matrix}\text{enhanced } L\text{-parameters}\\ (\varphi,\rho)\end{matrix}\right\}/\text{equiv.}$$

First step

G a p-adic group.

(1) We will choose a "color palette" $\mathfrak{B}(G)$ and paint the objects from the left hand side according to it in order that the objects of a given color $\mathfrak{s}$ have a nice structure (Harish-Chandra, Bernstein, Heiermann, Solleveld): they are in bijection with the simple modules of a twisted affine Hecke algebra $\mathcal{H}(G,\mathfrak{s})$.

A pictorial description II

Next steps

(2) We will choose a "color palette" $\mathfrak{B}^\vee(G)$ and paint the objects from the right hand side according to it in order that the objects of a given color $\mathfrak{s}^\vee$ have a nice structure (A.-Moussaoui-Solleveld): they are in bijection with the simple modules of a twisted affine Hecke algebra $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$.

(3) Expectation: We can choose the same color palette to paint the object from the right hand side "Cuspidality Conjecture" (A.-Moussaoui-Solleveld): $\mathfrak{s} \leftrightarrow \mathfrak{s}^\vee$.

(4) Expectation: The structures of both sides attached to the objects of the "same" color are closely related: when $\mathfrak{s} \leftrightarrow \mathfrak{s}^\vee$, the algebras $\mathcal{H}(G, \mathfrak{s})$ and $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$ are almost Morita equivalent (i.e. they have equivalent categories of finite dimensional representations).

Notation (group side)

- F a non-archimedean local field
- $\mathbf{G}$ connected reductive algebraic group defined over F
- G group of F-rational points of $\mathbf{G}$
- We suppose that G is a pure inner form of quasi-split group.

Notation (Galois side)

- W_F Weil group of F
- $G^\vee$ complex reductive group with root datum dual to that of G
- $\mathrm{Z}_{G^\vee}$ center of $G^\vee$
- $G^\vee_{\mathrm{der}}$ derived group of $G^\vee$

Definition

A Langlands parameter – or L-parameter – is a morphism $\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$ such

- $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is morphism of algebraic groups,
- $\varphi(w)$ is a semisimple element of $G^\vee$, for any $w \in W_F$.

Definition

- $\mathrm{Z}_{G^\vee_{\mathrm{der}}}(\varphi) := \mathrm{Z}_{G^\vee}(\varphi(W'_F))$, where $W'_F := W_F \times \mathrm{SL}_2(\mathbb{C})$
- An enhanced L-parameter is a pair (φ, ρ) where φ is an L-parameter for G and $\rho \in \mathrm{Irr}(\mathcal{S}_\varphi)$, with $\mathcal{S}_\varphi := \mathrm{Z}_{G^\vee_{\mathrm{der}}}(\varphi)/\mathrm{Z}_{G^\vee_{\mathrm{der}}}(\varphi)^\circ$.
- The representation ρ is called an enhancement of φ.
- Action of $G^\vee$ on the set of enhanced L-parameters:

$$g \cdot (\varphi, \rho) := (g\varphi g^{-1}, {}^g\rho), \ \ \text{for } g \in G^\vee,$$

  where ${}^g\rho\colon h \mapsto \rho(g^{-1}hg)$.
- $\Phi_{\mathrm{e}}(G)$ set of $G^\vee$-conjugacy classes of enhanced L-parameters.

The conjectural local Langlands correspondence

It is expected to be a "nice" bijection (i.e. satisfying a list of properties)

$$\mathrm{LLC}\colon \mathrm{Irr}(G) \xrightarrow{1-1} \Phi_{\mathrm{e}}(G),$$

where $\mathrm{Irr}(G)$ is the set of isomorphism classes of irreducible smooth representations of G.

Useful viewpoint

For φ a given L-parameter, we define

- $\mathcal{G}_\varphi := \mathrm{Z}_{G^\vee_{\mathrm{der}}}(\varphi|_{W_F})$ (a possibly disconnected complex reductive group); $\mathcal{G}_\varphi^\circ$ the identity component of $\mathcal{G}_\varphi$;
- $u_\varphi := \varphi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) \in \mathcal{G}_\varphi^\circ$ (a unipotent element);
- $A_{\mathcal{G}_\varphi}(u_\varphi) := \pi_0(\mathrm{Z}_{\mathcal{G}_\varphi}(u_\varphi))$.

We have $S_\varphi \simeq A_{\mathcal{G}_\varphi}(u_\varphi)$: it will allow to use the generalized Springer correspondence for $\mathcal{G}_\varphi$!

Generalized Springer variety [Lusztig, Invent. math. 1984]

- $\mathcal{P} = \mathfrak{L}\mathcal{U}$ parabolic subgroup of $\mathcal{G}^\circ$
- $u \in \mathcal{G}^\circ$ and $v \in \mathfrak{L}$ unipotent elements.

The group $\mathrm{Z}_{\mathcal{G}^\circ}(u) \times \mathrm{Z}_{\mathfrak{L}}(v)\mathcal{U}$ acts on the variety

$$Y_{u,v} := \left\{ y \in \mathcal{G}^\circ \,:\, y^{-1}uy \in v\mathcal{U} \right\}$$

by $(g,p) \cdot y := gyp^{-1}$, with $g \in \mathrm{Z}_{\mathcal{G}^\circ}(u)$, $p \in \mathrm{Z}_{\mathfrak{L}}(v)\mathcal{U}$ and $y \in Y_{u,v}$.

The group $A_{\mathcal{G}^\circ}(u) \times A_{\mathfrak{L}}(v)$ acts on the set of irreducible components of $Y_{u,v}$ of maximal dimension (i.e. $\dim \mathcal{U} + \frac{1}{2}(\dim \mathrm{Z}_{\mathcal{G}^\circ}(u) + \dim \mathrm{Z}_{\mathfrak{L}}(v))$). Let $\sigma_{u,v}$ denote the corresponding permutation representation.

Definition [Lusztig, Invent. math. 1984]

Let $\rho^\circ \in \mathrm{Irr}(A_{\mathcal{G}^\circ}(u))$. Then ρ° is called cuspidal if

$$\langle \rho^\circ, \sigma_{u,v} \rangle_{A_{\mathcal{G}^\circ}(u)} \neq 0 \text{ for any unipotent } v \in \mathfrak{L} \quad \Rightarrow \quad \mathcal{P} = \mathcal{G}^\circ,$$

where $\langle \; , \; \rangle_{A_{\mathcal{G}^\circ}(u)}$ is the usual scalar product on the space of class functions on $A_{\mathcal{G}^\circ}(u)$ with values in $\overline{\mathbb{Q}}_\ell$.

Remark

If (u, ρ) is cuspidal, then $\mathcal{C}$ is a *distinguished* (i.e. $\mathcal{G}^\circ$ does not meet the unipotent variety of $\mathfrak{L}$ for any $\mathfrak{L} \neq \mathcal{G}^\circ$). However, in general not every distinguished unipotent class supports a cuspidal representation.

Cuspidal pairs in arbitrary complex reductive groups

Let $\mathcal{G}$ be a possibly disconnected reductive group over $\mathbb{C}$, with identity component $\mathcal{G}^\circ$. Let $u \in \mathrm{U}(\mathcal{G})$ and $\rho \in \mathrm{Irr}(A_\mathcal{G}(u))$. We observe that $A_{\mathcal{G}^\circ}(u) \subset A_\mathcal{G}(u)$. The pair (u, ρ) is called cuspidal if the restriction of ρ to $A_{\mathcal{G}^\circ}(u)$ is a direct sum of irreducible representations ρ° such that one (or equivalently any) of the pairs (u, ρ°) is cuspidal.

Definition [A-Moussaoui-Solleveld (2015)]

An enhanced L-parameter $(\varphi, \rho) \in \Phi_\mathrm{e}$ is called cuspidal if the following properties hold:

- φ is discrete (i.e., $\varphi(W_F')$ is not contained in any proper Levi subgroup of $G^\vee$),
- (u_φ, ρ) is a cuspidal pair in $\mathcal{G}_\phi$.

Notation: $\Phi_{\mathrm{e,cusp}} :=$ $G^\vee$-conjugacy of cuspidal enhanced L-parameters.

Cuspidality Conjecture for LLC [A-Moussaoui-Solleveld]

The cuspidal enhanced Langlands parameters for G correspond by the LLC to the irreducible supercuspidal representations of G:

$$\mathrm{LLC}\colon \mathrm{Irr}_{\mathrm{cusp}}(G) \xrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(G). \tag{1}$$

The Cuspidality Conjecture is known to hold in the following cases:

- general linear groups and split classical p-adic groups [Moussaoui, 2017]
- inner forms of general linear groups and of special linear groups [A-Moussaoui-Solleveld, 2018]
- pure inner forms of quasi-split classical p-adic groups [A-Moussaoui-Solleveld, 2022]
- the exceptionnal p-adic group G_2 [A-Xu, 2022]
- unipotent supercuspidal representations of an arbitrary group G ([Lusztig, 1995] + [Feng-Opdam-Solleveld, 2020])

Supercuspidal support [Harish-Chandra]

Let π be an irreducible smooth representation of G.

1. There exists a parabolic subgroup P of G, with Levi factor L, and $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)$ such that π embeds in $\mathrm{i}_P^G \sigma$.
2. If P' is a parabolic subgroup of G, with Levi factor L', and $\sigma' \in \mathrm{Irr}_{\mathrm{cusp}}(L')$, then π is isomorphic to a subquotient of $\mathrm{i}_{P'}^G \sigma'$ if and only if there exists an element of G conjugating (L, σ) and (L', σ').
3. The G-conjugacy class $(L, \sigma)_G$ of (L, σ) is called the supercuspidal support of π.
4. We denote by Sc the map defined by

$$\mathrm{Sc}(\pi) := (L, \sigma)_G. \tag{2}$$

Slogan:

The Harish-Chandra "philosophy of cusp forms" still works on the Galois side of the LLC.

Notation

- L Levi subgroup of a parabolic subgroup P of G
- $\mathfrak{X}_{\mathrm{nr}}(L)$ group of unramified characters of L (a character is unramified if it is trivial on every compact subgroup of L)
- σ be irreducible supercuspidal smooth representation of L
- $\mathfrak{s} := [L, \sigma]_G$ for the G-conjugacy class of the pair $(L, \mathfrak{X}_{\mathrm{nr}}(L) \cdot \sigma)$
- $\mathfrak{B}(G)$ set of such classes $\mathfrak{s}$.

Definition

Let $\mathfrak{s} \in \mathfrak{B}(G)$. The Bernstein series attached to $\mathfrak{s}$ is the set $\mathrm{Irr}^{\mathfrak{s}}(G)$ of (isomorphism classes of) irreducible representations of G whose supercuspidal support lies in $\mathfrak{s}$.

Theorem [Bernstein]

We have a partition of $\mathrm{Irr}(G)$:

$$\mathrm{Irr}(G) = \bigsqcup_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Irr}^{\mathfrak{s}}(G). \tag{3}$$

Definiton

- $L^\vee$ Langlands dual group of L and $\iota_{L^\vee}\colon L^\vee \hookrightarrow G^\vee$ the canonical embedding
- $\mathfrak{X}_{\mathrm{nr}}(L^\vee) := \{\zeta\colon W_F/I_F \to \mathrm{Z}^\circ_{L^\vee}\}$, which acts on the set of cuspidal enhanced L-parameters for L.
- $\mathfrak{s}^\vee := [L^\vee \rtimes W_F, (\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}})]_{G^\vee}$ the $G^\vee$-conjugacy class of $(L^\vee \rtimes W_F, \mathfrak{X}_{\mathrm{nr}}(L^\vee)\cdot(\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}}))$, where $(\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}}) \in \Phi_{\mathrm{e,cusp}}(L)$
- $\mathfrak{B}^\vee(G)$ the set of such $\mathfrak{s}^\vee$.

Theorem [Lusztig, Achar-Henderson-Juteau-Riche]

Let $\mathcal{G}^\circ$ be a connected complex reductive group, $\mathcal{C} \in \mathrm{Unip}(\mathcal{G}^\circ)$ and $\mathcal{E}$ irreducible $\mathcal{G}^\circ$-equivariant local system on $\mathcal{C}$. Then $\mathcal{F}_\rho := \mathrm{IC}(\mathcal{C}, \mathcal{E}_\rho)$ occurs as a summand of $\mathrm{i}_{\mathfrak{L}\subset\mathcal{P}}^{\mathcal{G}^\circ}(\mathrm{IC}(\mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}}))$, for some triple $(\mathcal{P}, \mathfrak{L}, (\mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}}))$, where $\mathcal{P}$ is a parabolic subgroup of $\mathcal{G}^\circ$ with Levi subgroup $\mathfrak{L}$ and $(\mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}})$ is a cuspidal unipotent pair in $\mathfrak{L}$. Moreover, the triple $(\mathcal{P}, \mathfrak{L}, \mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}})$ is unique up to $\mathcal{G}^\circ$-conjugation.

The cuspidal support map

Let $\rho^\circ \in \mathrm{Irr}(A_{\mathcal{G}^\circ}(u))$. The cuspidal support of (u, ρ°) is

$$\mathrm{Sc}^{\mathcal{G}^\circ}(u, \rho^\circ) := (\mathfrak{L}, (v, \rho^\circ_{\mathrm{cusp}}))_{\mathcal{G}^\circ}, \quad \text{where } v \in \mathcal{C}_{\mathrm{cusp}} \text{ and } \rho^\circ_{\mathrm{cusp}} \leftrightarrow \mathcal{E}_{\mathrm{cusp}}.$$

The case of disconnected groups

We set $\mathcal{T} := \mathrm{Z}^\circ_{\mathfrak{L}}$ and $\mathcal{M} := \mathrm{Z}_{\mathcal{G}}(\mathcal{T})$. The cuspidal support of (u, ρ) is a (well-defined) triple $(\mathcal{M}, v, \rho_{\mathrm{cusp}})_{\mathcal{G}}$, where $\rho^\circ_{\mathrm{cusp}}$ occurs in the restriction of ρ_{cusp} to $A_{\mathcal{G}^\circ}(u)$.

Application to enhanced L-parameters

Let $(\varphi, \rho) \in \Phi_{\mathrm{e}}(G)$. We write $u_\varphi := \varphi\left(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)$. Upon conjugating φ with a suitable element of $\mathrm{Z}_{\mathcal{G}_\varphi^\circ}(u_\varphi)$, we may assume that $\mathfrak{L}$ contains $\varphi\left(\left(1, \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)\right)\right)$ for all $z \in \mathbb{C}^\times$.

By the Jacobson–Morozov theorem, any unipotent element v of $\mathfrak{L}$ can be extended to a homomorphism of algebraic groups

$$j_v \colon \mathrm{SL}_2(\mathbb{C}) \to \mathfrak{L} \text{ satisfying } j_v \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = v. \tag{4}$$

Moreover, this extension is unique up to conjugation in $\mathrm{Z}_{\mathfrak{L}}(v)^\circ$. A homomorphism j_v satisfying these conditions to be *adapted to* φ.

Remark

Up to $G^\vee$-conjugacy, there exists a unique homomorphism $j_v \colon \mathrm{SL}_2(\mathbb{C}) \to \mathfrak{L}$ which is adapted to φ, and moreover, the cocharacter

$$\chi_{\varphi,v} \colon z \mapsto \varphi\left(1, \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)\right) \cdot j_v \left(\begin{smallmatrix} z^{-1} & 0 \\ 0 & z \end{smallmatrix}\right) \tag{5}$$

has image in $\mathcal{T}$.

Supercuspidal support on Galois side

We define an L-parameter $\varphi_v \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Z}_{G^\vee}(\mathcal{T})$ by

$$\varphi_v(w, x) := \varphi(w, 1) \cdot \chi_{\varphi,v}(\|w\|^{1/2}) \cdot j_v(x) \quad \text{for all } w \in W_F,\ x \in \mathrm{SL}_2(\mathbb{C}).$$

The cuspidal support of (φ, ρ) is defined to be

$$\mathrm{Sc}(\varphi, \rho) := (\mathrm{Z}_{G^\vee}(\mathcal{T}), (\varphi_v, \rho_{\mathrm{cusp}})). \tag{6}$$

Definition

Let $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$ be the fiber of $\mathfrak{s}^\vee$ under the map Sc.

Theorem [A.-Moussaoui-Solleveld]

The set $\Phi_{\mathrm{e}}(G)$ of $G^\vee$-conjugacy classes of enhanced L-parameters is partitioned into *series à la Bernstein* as

$$\Phi_{\mathrm{e}}(G) = \coprod_{\mathfrak{s}^\vee \in \mathfrak{B}(G^\vee)} \Phi^{\mathfrak{s}^\vee}(G). \tag{7}$$

Theorem [A.-Moussaoui-Solleveld]

For every $\mathfrak{s}^\vee \in \mathfrak{B}(G^\vee)$, there is twisted extended affine Hecke algebra $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$ such that

$$\mathrm{Irr}(\mathcal{H}(G^\vee, \mathfrak{s}^\vee)) \xleftrightarrow{1-1} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G).$$

A variation of $\mathcal{G}_\varphi$:

Let $I_F \subset W_F$ be the inertia group of F. We define

$$J_\varphi := \mathrm{Z}_{G^\vee}(\varphi(I_F)).$$

A root system:

Define $R(J^\circ, \mathcal{T})$ to be the set of $\alpha \in X^*(\mathcal{T})\backslash\{0\}$ which appear in the adjoint action of $\mathcal{T}$ on the Lie algebra of J_φ°. It can be shown that $R(J^\circ, \mathcal{T})$ is a root system. We denote by $W_{\mathfrak{s}^\vee}^\circ$ its Weyl group.

An extended finite Weyl group:

Let $W_{\mathfrak{s}^\vee} := \mathrm{N}_{G^\vee}(\mathfrak{s}^\vee)/L^\vee$. We have $W_{\mathfrak{s}^\vee} = W^\circ_{\mathfrak{s}^\vee} \rtimes \mathfrak{R}_{\mathfrak{s}^\vee}$, where

$$\mathfrak{R}_{\mathfrak{s}^\vee} := \left\{ w \in W_{\mathfrak{s}^\vee} \,:\, w(R(J^\circ, \mathcal{T})^+) \subset R(J^\circ, \mathcal{T})^+ \right\}.$$

A root datum:

We define a root datum

$$\mathcal{R}_{\mathfrak{s}^\vee} := (R_{\mathfrak{s}^\vee}, X^*(T_{\mathfrak{s}^\vee}), R^\vee_{\mathfrak{s}^\vee}, X_*(T_{\mathfrak{s}^\vee})),$$

where $T_{\mathfrak{s}^\vee} \simeq \mathfrak{s}^\vee_L = [L^\vee, (\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}})]_{L^\vee}$ and

$$R_{\mathfrak{s}^\vee} = \{ m_\alpha \, \alpha \,:\, \alpha \in R(J^\circ, \mathcal{T})_{\mathrm{red}} \subset X^*(T_{\mathfrak{s}^\vee}) \},$$

with $m_\alpha \in \mathbb{Z}_{>0}$. The group $W_{\mathfrak{s}^\vee}$ acts on $\mathcal{R}_{\mathfrak{s}^\vee}$.

Weight functions:

We define $W_{\mathfrak{s}^\vee}$-invariant functions

$$\lambda\colon R_{\mathfrak{s}^\vee} \to \mathbb{Q}_{>0} \quad \text{and} \quad \lambda^*\colon \{m_\alpha \alpha \in R_{\mathfrak{s}^\vee} : m_\alpha \alpha \in 2X_*(T_{\mathfrak{s}^\vee})\} \to \mathbb{Q}.$$

A twisted affine Hecke algebra:

The algebra $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$ is defined as

$$\mathcal{H}(G^\vee, \mathfrak{s}^\vee) := \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, z) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \natural_{\mathfrak{s}^\vee}],$$

where $\natural_{\mathfrak{s}^\vee}$ is a certain 2-cocycle.

Remark

In applications to p-adic groups, we will take $z = q^{1/2}$, where q is the cardinality of the residual field of F.

General spin groups

Take $G = G_n = \mathrm{GSpin}_N(F)$ assumed to be split (for simplicity). Then

$$\mathrm{GSpin}_{2n+1}^\vee = \mathrm{GSp}_{2n}(\mathbb{C}) \text{ and } \mathrm{GSpin}_{2n}^\vee = \mathrm{GSO}_{2n}(\mathbb{C}).$$

Notation:

- $\mu_{G^\vee} \colon G^\vee \to \mathbb{C}^\times$ the similitude character.
- $\iota \colon G^\vee \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$ standard embedding
- For $a \in \mathbb{Z}_{>0}$, let P_a be the unique irred. repr. of $\mathrm{SL}_2(\mathbb{C})$ of dimension a.

Let $\varphi \in \Phi(G)$. We set

$$\mathrm{Jord}(\varphi) := \{(\tau, a) \in \mathrm{Irr}(W_F) \times \mathbb{Z}_{>0} \, : \, \tau \otimes P_a \text{ occurs in } \iota \circ \varphi\},$$

and for $\tau \in \mathrm{Irr}(W_F)$:

$$\mathrm{Jord}_\tau(\varphi) := \{a \in \mathbb{Z}_{>0} \, : \, (\tau, a) \in \mathrm{Jord}(\varphi)\}.$$

Every standard Levi subgroup of G is of the form:

$$L = G_{n_-} \times \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F), \quad \text{where } n_- + n_1 + \cdots + n_r = n.$$

We have

$$L^\vee = G_{n_-}^\vee \times \mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{C}).$$

Let $\iota_{L^\vee} : L^\vee \hookrightarrow \mathrm{GL}_{2n_-}(\mathbb{C}) \times \mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{C})$ be the standard embedding and let $\varphi_L \in \Phi(L)$. Then

$$\iota_{L^\vee} \circ \varphi_L = \varphi_- \oplus \bigoplus_i (\varphi_i \oplus \varphi_i^\vee),$$

where $\varphi_- \in \Phi(G_{n_-})$ and $\varphi_i \in \Phi(\mathrm{GL}_{n_i}(F))$.
We have $S_{\varphi_L} = S_{\varphi_-}$ because the $\mathrm{Z}_{\mathrm{GL}_{n_i}(\mathbb{C})}(\varphi_i)$ are connected.

Cuspidality

Let $\rho \in \mathrm{Irr}(S_\varphi) = \mathrm{Irr}(S_{\varphi_-})$. We have

$$\rho \text{ cuspidal } \Leftrightarrow \begin{cases} (\varphi_-, \rho) \text{ is cuspidal} \\ \text{the } (\varphi_i, \mathrm{triv}) \text{ are cuspidal} \end{cases}.$$

- $(\varphi_i, \mathrm{triv})$ cuspidal $\Leftrightarrow$ $(\varphi_i|_{\mathrm{SL}_2(\mathbb{C})} = 1$ and $\varphi_i \in \mathrm{Irr}(W_F))$.
- (φ_-, ρ) is cuspidal iff
  - $\mathrm{Jord}(\varphi)$ has no hole, i.e. if $(\tau, a) \in \mathrm{Jord}(\varphi)$ and $a > 2$, then also $(\tau, a-2) \in \mathrm{Jord}(\varphi)$;
  - ρ is "alternated" as defined by Mœglin.

We define

$$\mathrm{Irr}(W_F)^{\pm}_{\varphi} := \{\tau \in \mathrm{Irr}(W_F) \,:\, \tau \simeq \tau^{\vee} \otimes \mu_{G^{\vee}} \circ \varphi,\, \mathrm{sgn}(\tau) = \pm \mathrm{sgn}({G^{\vee}}_{\mathrm{der}})\}.$$

and

$$a_\tau := \begin{cases} \max \mathrm{Jord}_\tau(\varphi_-) & \text{if } \mathrm{Jord}_\tau(\varphi_-) \neq \emptyset \\ 0 & \text{if } \mathrm{Jord}_\tau(\varphi_-) = \emptyset \text{ and } \tau \in \mathrm{Irr}(W_F)^-_\varphi \\ -1 & \text{if } \mathrm{Jord}_\tau(\varphi_-) = \emptyset \text{ and } \tau \in \mathrm{Irr}(W_F)^+_\varphi. \end{cases}$$

The weight functions

The root system $R_{\mathfrak{s}^\vee}$ is a disjoint union of root subsystems $R_{\mathfrak{s}^\vee,\tau}$.

- If $\alpha \in R_{\mathfrak{s}^\vee,\tau,\mathrm{red}}$ is a short root in a type B root system, then $m_\alpha = t_\tau$ (torsion number of τ),

$$\lambda(\alpha) = t_\tau(a_\tau + a_{\tau'} + 2)/2 \quad \text{and} \quad \lambda^*(\alpha) = t_\tau(a_\tau - a_{\tau'})/2.$$

- Otherwise, $\lambda(\alpha) = \lambda^*(\alpha) = m_\alpha$, and equals either t_τ or $t_\tau/2$.

Theorem [Heiermann, Solleveld]

For every $\mathfrak{s} \in \mathfrak{B}(G)$, there is twisted extended affine Hecke algebra $\mathcal{H}(G, \mathfrak{s})$ such that

$$\mathrm{Irr}(\mathcal{H}(G, \mathfrak{s})) \overset{1-1}{\leftrightarrow} \mathrm{Irr}^{\mathfrak{s}}(G).$$

Theorem [A-Moussaoui-Solleveld, 2022]

When G is a pure inner form of a quasi-split p-adic classical group ($\mathrm{Sp}_{2n}(F)$, $\mathrm{SO}_N(F)$, $\mathrm{O}_N(F)$, $\mathrm{GSpin}_N(F)$, $\mathrm{U}_N(F)$), there is an isomorphism

$$\mathcal{H}(G, \mathfrak{s}) \simeq \mathcal{H}(G^\vee, \mathfrak{s}^\vee),$$

the cocycles $\natural_{\mathfrak{s}}$ and $\natural_{\mathfrak{s}^\vee}$ are both trivial, and the induced bijection

$$\mathfrak{L}_G : \mathrm{Irr}(G) \to \Phi_{\mathrm{e}}(G)$$

coincides with the LLC constructed by Arthur.

An example with non-trivial cocycle [A-Baum-Plymen-Solleveld]

- $\varphi \in \Phi(\mathrm{GL}_2(F))$ whose kernel contains a Frobenius element of W_F and with image $\{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) : a, b \in \{\pm 1\}\}$
- $\varphi \overset{\mathrm{LLC}}{\longleftrightarrow} \pi \in \mathrm{Irr}(\mathrm{GL}_2(F))$ with $\mathfrak{X}(\pi) = \{1, \gamma, \eta, \gamma\eta\}$ (unramified characters of order ≤ 2)
- D central division over F of dimension 4, $\tau \in \mathrm{Irr}(D^\times) \overset{\mathrm{JL}}{\longleftrightarrow} \pi$
- $G := \mathrm{SL}_5(D)$, $L := G \cap \mathrm{GL}_1(D)^5$, $\mathfrak{s} = [L, \sigma]_G$, where σ occurs in the restriction to L of $\tau \otimes 1 \otimes \gamma \otimes \eta \otimes \gamma\eta \in \mathrm{Irr}(\mathrm{GL}_1(D)^5)$
- The algebras $\mathcal{H}(G, \mathfrak{s})$ and $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$ are not Morita equivalent, but their categories of finite dimensional modules are equivalent.

Theorem [A-Moussaoui-Solleveld]

Let G be any inner form of $\mathrm{SL}_n(F)$ and let $\mathfrak{s} \in \mathfrak{B}(G)$. The categories of finite dimensional modules of $\mathcal{H}(G, \mathfrak{s})$ and $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$ are equivalent, in particular we have a bijection

$$\mathrm{Irr}(\mathcal{H}(G, \mathfrak{s})) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{H}(G^\vee, \mathfrak{s}^\vee)).$$

Thank you very much for your attention.