BZSV Duality and Relative Trace Formula

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1. Background on period integrals
2. BZSV duality conjecture
3. Relative trace formula

An example

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g the corresponding level 4 weight $k + \frac{1}{2}$ form, in Kohnen plus space.

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There are two aspects of this result:
1. Fourier coefficient of $g(z)$ is related to $L-$value attached to g.
2. The correspondence between f and g is a case of Langlands functoriality, $L-$value of g equals to L-value of f.

Gan-Gross-Prasad, Ichino-Ikeda conjecture

If π_1, π_2 are irreducible automorphic representations of SO_n and SO_m with $n > m$, let $\phi_1 \in \pi_1, \phi_2 \in \pi_2$, then

$$\frac{|\int_{[\mathrm{SO}_m]} \mathcal{F}\phi_1(g)\phi_2(g)\ dg|^2}{\|\phi_1\|^2\|\phi_2\|^2} \sim \frac{L(\frac{1}{2}, \pi_1 \times \pi_2)}{L(1, \pi_1, Ad)L(1, \pi_2, Ad)}.$$

Here $\sim$ means the equality holds up to some local integrals over finitely many local places and a global factor in a fixed finite set.
$\mathcal{F}\phi_1$ is a Fourier coefficient of ϕ_1:

$$\mathcal{F}\phi_1(g) = \int_{[U]} \phi_1(ug)\psi^{-1}(u)\ du$$

for a suitable unipotent subgroup U of SO_n and a suitable ψ.

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Main progress: For unitary group, the conjecture is known for $n = m + 2d + 1$.

Relative trace formula: the method used to prove Gan-Gross-Prasad for unitary group.
Work by W. Zhang, Z.Yun, Beuzart-Plessis-Chaudouard (2025) based on a relative trace formula conjectured by Jacquet-Rallis:
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Rest of the talk:
Describe the duality conjecture of Ben-Zvi-Sakellaridis-Venkatesh, which is about both functoriality and period integral.
Discuss a relative trace formula approach to BZSV duality.

BZSV duality is about two dual quadruples Δ and $\hat{\Delta}$.
$\Delta = (G, H, \rho_H, \iota)$ where

- G is a split reductive group
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- ι is a homomorphism $\mathrm{SL}_2 \mapsto G$ whose image commutes with H.

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The period integral attached to Δ is (for $\phi \in \pi$ on G)

$$\mathcal{P}_\Delta(\phi, \Theta) = \int_{[H]} P_\iota(\phi)(h)\Theta(h)\, dh.$$

Here P_ι is a Fourier coefficient of ϕ determined by ι, and Θ is in a space of theta functions on H determined by ρ_H.

BZSV duality conjecture 2025+

Given $\Delta = (G, H, \rho, \iota)$, there is $\hat{\Delta} = (G', H', \rho', \iota')$, so that

- $G' = \hat{G}$ is the Langlands dual of G.

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- If $\mathcal{P}_\Delta(\phi, \Theta)$ is nonvanishing, then π is a functorial lift from a representation Π of $\hat{H}'$ (the dual of H'). (More precisely, the Arthur parameter of π factors through $\iota' : H' \times \mathrm{SL}_2 \mapsto \hat{G}$.)

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- $$\frac{|\mathcal{P}_\Delta(\phi, \Theta)|^2}{\|\phi\|^2 \|\Theta\|^2} \sim \frac{L(\Pi, \rho', \iota')}{L(1, \Pi, Ad)^2}.$$

  Here $L(\Pi, \rho', \iota')$ is a product of values of $L-$functions of Π determined by ρ' and ι'. These are not necessarily central values of the $L-$function.

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  Here $L(\Pi, \rho', \iota')$ is a product of values of $L-$functions of Π determined by ρ' and ι'. These are not necessarily central values of the $L-$function.
- The dual assertions hold for the period $\mathcal{P}_{\hat{\Delta}}(\phi', \Theta')$ on G'.

A motivating example

When $\Delta = (\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2n}, 0, 1)$, its dual is

$$\hat{\Delta} = (\mathrm{Sp}_{2n} \times \mathrm{SO}_{2n}, \mathrm{Sp}_{2n} \times \mathrm{SO}_{2n}, std, 1)$$

(std denotes the standard representation $\mathrm{Sp}_{2n} \times \mathrm{SO}_{2n} \mapsto \mathrm{Sp}_{4n^2}$).

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The period integral $\mathcal{P}_{\hat{\Delta}}$ is a co-period: a pairing over the whole group of an automorphic form and a theta function.
The pairing is nonzero only when the representation $\pi \times \tau$ on $\mathrm{Sp}_{2n} \times \mathrm{SO}_{2n}$ is such that π and τ correspond under theta correspondence. Thus $\pi \times \tau$ is a functorial lift from τ on SO_{2n}.
The identity for $\mathcal{P}_{\hat{\Delta}}(\phi', \Theta')$ follows from Rallis inner product formula for theta correspondence.

Strongly tempered BZSV quadruple

A quadruple Δ is strongly tempered if its dual $\hat{\Delta}$ has the form $(\hat{G}, \hat{G}, \hat{\rho}, 1)$.

Another example

$\Delta = (\mathrm{GL}_6 / Z, \mathrm{GL}_2 / Z, 0, \iota)$ and $\hat{\Delta} = (\mathrm{SL}_6, \mathrm{SL}_6, \wedge^3, 1)$.
Here let $P = MN$ be a standard parabolic in GL_6 / Z with $M = \mathrm{GL}_2^3 / Z$. The centralizer of the image of ι is in M and isomorphic to $H = \mathrm{GL}_2 / Z$.

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The period integral $\mathcal{P}_\Delta$ is Ginzburg-Rallis period, with known relation to exterior cube $L-$value.
As $\wedge^3 : \mathrm{SL}_6 \mapsto \mathrm{Sp}_{20}$ actually factors through $\mathrm{SL}_6 \mapsto \widetilde{\mathrm{Sp}}_{20}$, the theta functions on $\widetilde{\mathrm{Sp}}_{20}$ restricts to SL_6. The coperiod integral $\mathcal{P}_{\hat{\Delta}}$ has the form:

$$\int_{[\mathrm{SL}_6]} \phi(g)\Theta(\wedge^3(g))\, dg.$$

Question: Can we use RTF to study the corresponding co-periods $\mathcal{P}_{\hat{\Delta}}(\phi, \Theta)$?

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Mao-Rallis, 1997

There is a relative trace formula identity relating these coperiods to Fourier coefficients on SL_2.

Relative trace formula: an example

Let f be a Schwartz function on $\mathrm{SL}_6(A_k)$, (A_k adele ring over number field k). Define kernel function

$$K_f(x,y) = \sum_{\gamma \in \mathrm{SL}_6(k)} f(x^{-1}\gamma y)$$

Let $U = \{u(X) = \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\} \subset \mathrm{SL}_6$ where X are 3×3 matrices.

Define distribution

$$I(f) = \int_{[\mathrm{SL}_6]} \int_{[U]} K_f(g,u)\Theta(\wedge^3(g))\psi_U(u)\; du\; dg$$

where $\psi_U(u(X)) = \psi(-Tr(X))$.

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Then there is a *correspondence* between the Schwartz functions f on SL_6 and the Schwartz functions f' on SL_2 such that the relative trace identity $I(f) = J(f')$ holds, where $J(f')$ is the Kuznetsov trace formula on SL_2:

$$J(f') = \int_{[N]} \int_{[N]} K_{f'}(n_1, n_1)\psi(n_1^{-1}n_2)\ dn_1\ dn_2.$$

Implication of RTF

From the spectral decomposition of RTF:

- If coperiod $\langle \phi, \Theta(\wedge^3 \cdot) \rangle$ is nonzero for $\phi \in \pi$, then π is in the image of functorial lift from SL_2.

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- the product of the coperiod with the (U, ψ_U) coefficient on SL_6 equals the square of Fourier coefficient on SL_2.
- if we know the relation between Fourier coefficients and $L-$values, we can derive a relation between coperiod and $L-$value (proving the identity predicted in BZSV conjecture.)

Background on this work:

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- We were stuck for couple of months trying to guess the form of the relative trace formula identity for other cases. (Guess U and ψ_U)
- We tried to extend the work to other coperiod integrals but did not know how.

Motivation in 1995

Our guess in the G_2 case is a comparison of RTFs on SL_2 and on cubic cover of SL_2. It is based on

Duke-Iwaniec 1993

If F is a finite field of size $q = p^r \equiv 1 \bmod 3$. Let $\psi(x) = e^{2\pi i T(x)/p}$ and χ be a multiplicative character of order 3 on F^*, then

$$\sum_{x \in F} \psi(x^3 - 3x) = \sum_{x \in F^*} \chi(x)\psi(x + \frac{1}{x})$$

The RHS is a Salié sum (twisted Kloosterman sum), which appears in orbital integrals of Kuznetsov trace formula on covering groups.
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The LHS determines ψ_U (which is roughly $x \mapsto \psi(-3x)$.)
Rallis later pointed out the connection with Jordan algebra, then it was clear that $3x$ is the trace of xI_3, and the choice of ψ_U in general is $x \mapsto \psi(-T(x))$.

Generalization

Looking back, while Rallis was fully aware of Knop's work on spherical varieties, we did not know Hamiltonian space.
Given the dual quadruple $\Delta = (\mathrm{GL}_6 / Z, \mathrm{GL}_2 / Z, 0, \iota)$, we can see that U and ψ_U are determined by ι, through a duality on nilpotent orbits by Barbasch-Vogan (1985). The RTF comparison is between the distributions on SL_6 and SL_2 (Langlands dual of GL_2 / Z.)

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A conjecture, Mao-Wan-Lei Zhang

Let $\Delta = (G, H, \rho_H, \iota)$ be a BZSV quadruple. We conjecture a RTF comparison between $\hat{G}$ and $\hat{H}$ (dual of H) which reflects the functorial lift from $\hat{H}$ to $\hat{G}$.
Let $\hat{\Delta}$ be the dual of Δ. Let Δ' be the quadruple dual to $(H, H, \rho_\Delta, 1)$ (where ρ_Δ can be determined from ρ_H and ι with an explicit algorithm, $\rho_\Delta = \rho_H \oplus \cdots$).

- Let $I(f)$ on $\hat{G}$ be the distribution defined by $\mathcal{P}_{\hat{\Delta}}$ and (U, ψ_U)-coefficient (determined by ι).
- Let $J(f')$ on $\hat{H}$ be the distribution defined by $\mathcal{P}_{\Delta'}$ and $(N_{\hat{H}}, \psi)$-coefficient (generic Whittaker coefficient on $\hat{H}$.)
- We expect a relative trace identity $I(f) = J(f')$.

Algorithm for ρ_Δ:

- ι induces homomorphism $H \times \mathrm{SL}_2 \mapsto G$
- This induces the adjoint action of $H \times \mathrm{SL}_2$ on the Lie algebra of G which decomposes as a representation:
$$\oplus_{k \in I} \rho_k \otimes Sym^k$$
with a finite index set I.
- When k is odd ρ_k is symplectic. Let
$$\rho' = \oplus_{k \text{ odd} \in I} \rho_k.$$
- Let $\{\sigma_j | j \in J\}$ be the set of irreducible symplectic representations appearing in ρ' odd amount of times. Then $\rho_\Delta = \rho_H \oplus_{j \in J} \sigma_j$.

Motivation: If we accept BZSV duality conjecture and an extension of period integral conjecture to (U, ψ_U) coefficients (degenerate Whittaker periods), then the same $L-$values appear on spectral sides of the relative trace formulas.

This conjecture can reduce the question on $\mathcal{P}_{\hat{\Delta}}$ to a question on $\mathcal{P}_{\Delta'}$, where Δ' is a strongly tempered BZSV quadruple.

Knop gave a classification of quadruples $(H, H, \rho, 1)$. We can write down the corresponding duals Δ', and the period integrals $\mathcal{P}_{\Delta'}$. Most of these periods have been studied before.

Evidence

- When $\hat{\Delta}$ is strongly tempered, (that is $\Delta = (G, G, \rho, 1)$), then the corresponding RTF identity is a trivial identity.
- RTF identity established for all cases when Δ is strongly tempered and H is of rank one. (Mao-Rallis and Mao-Wan-Zhang. ρ_Δ is trivial in these cases. $J(f')$ is Kuznetsov trace formula on $\hat{H}$)

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- When $\hat{\Delta} = (\mathrm{GL}_{2n}, \mathrm{GL}_n, 0, \iota')$ where $\iota'(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix})$, the dual quadruple is $\Delta = (\mathrm{GL}_{2n}, \mathrm{Sp}_{2n}, 0, 1)$. The conjecture is due to Friedberg-Jacquet (1995) comparing relative trace formula on GL_{2n} and Kuznetsov trace formula on SO_{2n+1}. The fundamental lemma is proved (Friedberg-Jacquet for unit element) when $n = 2$.

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- Switch the role of Δ and $\hat{\Delta}$, the relative trace identity compares a Kuznetsov trace formula on GL_n and a relative trace formula on GL_{2n}. It is established by Jacquet-Rallis (1993).

Friedberg-Jacquet:

$$I(f) = \int_{[S]} \int_{[N]} K_f(s,n)\psi_S(s)\psi_N(n)\ ds\ dn$$

where (N, ψ_N) is Whittaker coefficient, $S = \{s(g,v) = \begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix}\}$ with $\psi_S(s(g,v)) = \psi(T(v))$. Compared with Kuznetsov on odd orthogonal group.

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where (N, ψ_N) is Whittaker coefficient, $S = \{s(g,v) = \begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix}\}$ with $\psi_S(s(g,v)) = \psi(T(v))$. Compared with Kuznetsov on odd orthogonal group.

Jacquet-Rallis (the other one)

$$I(f) = \int_{[Sp_{2n}]} \int_{[N]} K_f(s,n)\psi_N(n)\ ds\ dn$$

where ψ_N is a degenerate character on N: $\psi_N(\begin{pmatrix} n_1 & v \\ & n_2 \end{pmatrix}) = \psi(n_1 n_2)$ with ψ a generic character.
Compared with Kuznetsov on GL_n.

More Examples

- When $\Delta = (\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2n}, 0, 1)$ and $\hat{\Delta} = (\mathrm{Sp}_{2n} \times \mathrm{SO}_{2n}, \mathrm{Sp}_{2n} \times \mathrm{SO}_{2n}, std, 1)$. The comparison is between RTF on $\mathrm{Sp}_{2n} \times \mathrm{SO}_{2n}$ and on SO_{2n} and can be proved using theta correspondence.

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- When $\Delta = (\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2n}, 0, 1)$ and $\hat{\Delta} = (\mathrm{Sp}_{2n} \times \mathrm{SO}_{2n}, \mathrm{Sp}_{2n} \times \mathrm{SO}_{2n}, std, 1)$. The comparison is between RTF on $\mathrm{Sp}_{2n} \times \mathrm{SO}_{2n}$ and on SO_{2n} and can be proved using theta correspondence.
- When $\hat{\Delta} = (\mathrm{GL}_n, \mathrm{GL}_{n-1} \times \mathrm{GL}_1, 0, 1)$ the dual quadruple is $\Delta = (\mathrm{GL}_n, \mathrm{GL}_2, 0, \iota)$. The comparison of relative traces is between GL_n and GL_2 which was conjectured by Mao-Rallis (1998). The proof can be given using theta correspondence. (Switch the role of Δ and $\hat{\Delta}$ gives a relative trace formula comparison that gives a lifting from $GL_{n-1} \times \mathrm{GL}_1$ to GL_n which is just Eisenstein series construction.)
  When $n = 3$, the RTF was studied in my Compte Rendu note in 1992.