HC modules & quantizations.

1) Quantization of singular symplectic varieties
2) Quantization of singular lagrangian subvarieties.

1) Quantization of singular symplectic varieties

Setting: $A=\bigoplus_{i\geqslant 0} A_i$ fin. gen'd commutative graded algebra$/\mathbb{C}$

$\mathcal{A}=\bigcup_{i\geqslant 0}\mathcal{A}_{\leqslant i}$ filtered assoc. algebra w. $\operatorname{gr}\mathcal{A}\xrightarrow{\sim} A$

$\Rightarrow [\mathcal{A}_{\leqslant i},\mathcal{A}_{\leqslant j}]\subset \mathcal{A}_{\leqslant i+j-1}\ \forall i,j \rightsquigarrow$ Poisson bracket on $\operatorname{gr}\mathcal{A}$ by

$$\{a+\mathcal{A}_{\leqslant i-1}, b+\mathcal{A}_{\leqslant j-1}\}:=[a,b]+\mathcal{A}_{\leqslant i+j-2}$$

If A already had Poisson structure and the isomorphism is Poisson, say $\mathcal{A}$ is a (filtered) quantization of A

Set $X:=\operatorname{Spec}(A)$ and <u>assume</u>:

(a) $A_0=\mathbb{C}$

(b) X is "singular symplectic" (a number of conditions incl. X^{reg} is symplectic & X is normal $\Rightarrow \operatorname{codim}_X X^{sing}\geqslant 2$)

Examples:

− of singular symplectic varieties.

$\mathfrak{g}$ is s/simple Lie algebra, $\mathbb{O} \subset \mathfrak{g}^*$ is nilpotent orbit
$A := \mathbb{C}[\mathbb{O}]$, $X :=$ normalization of $\overline{\mathbb{O}}$ w. $\{\cdot,\cdot\} \leftsquigarrow$ Kirillov-Kostant form, grading: from $\mathbb{C}^\times \curvearrowright \overline{\mathbb{O}}$ by rescaling.

Important generalization:

$A := \mathbb{C}[\tilde{\mathbb{O}}]$, $\tilde{\mathbb{O}}$ is G-equivariant cover of $\mathbb{O}$.

- of quantizations: $\mathbb{O} \subset \mathfrak{g}^*$ is principal $\rightsquigarrow X = \mathcal{N}$, nilpotent cone.
Center of $\mathcal{U}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W$; $\lambda \in \mathfrak{h}^*/W \rightsquigarrow$ max. ideal $m_\lambda \subset$ center
$\rightsquigarrow \mathcal{A}_\lambda = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})m_\lambda$ is a filt. quantization of $A = \mathbb{C}[X]$.

Thm (I.L. 2016): Under (a) & (b), have fin. dim. vector space $\mathfrak{h}_X$ w. action of finite group W_X s.t.

$$\begin{array}{ccc} \{\text{filtered quantizations of } A\}/\text{iso} & \xrightarrow{\sim} & \mathfrak{h}_X/W_X \\ \Psi & & \Psi \\ \mathcal{A}_\lambda & \longleftarrow\!\shortmid & \lambda \end{array}$$

Example con't: $\mathfrak{h}_X = \mathfrak{h}^*$, $W_X = W$, & $\mathcal{A}_\lambda := \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})m_\lambda$

Remarks: 1) When $A = \mathbb{C}[\mathbb{O}]$ (or $\mathbb{C}[\tilde{\mathbb{O}}]$), G acts on any quantization w. distinguished "quantum comoment map" $\mathcal{U}(\mathfrak{g}) \to \mathcal{A}$

– in the example above, this is the projection $\mathcal{U}(\mathfrak{g}) \twoheadrightarrow \mathcal{A}_\lambda$.
In particular, any $\mathcal{A}$-module is also a $\mathcal{U}(\mathfrak{g})$-module.

2) The quantization $\mathcal{A}_0$ is called canonical. For $A = \mathbb{C}[\tilde{\mathbb{O}}]$, $\ker[\,\mathcal{U}(\mathfrak{g}) \to \mathcal{A}_0\,]$ is called the unipotent ideal associated to $\tilde{\mathbb{O}}$, they generalize special unipotent ideals of Barbasch & Vogan (LMBM'21). From here we get definitions of "unipotent HC modules" in a number of cases.

2) Quantization of singular lagrangian subvarieties.

We'd like to understand certain simple modules $\mathcal{M}$ over quantizations $\mathcal{A}(=\mathcal{A}_\lambda)$ of $\mathbb{C}[X]$.

- Good filtrations: $\mathcal{M}$ is fin. gen'd $\mathcal{A}$-module. A good filtration on $\mathcal{M}$ is a filtration $\mathcal{M} = \bigcup_{j\geq 0} \mathcal{M}_{\leq j}$ s.t.
  - $\mathcal{A}_{\leq i}\mathcal{M}_{\leq j} \subset \mathcal{M}_{\leq i+j}$
  - $\mathrm{gr}\,\mathcal{M}$ is fin. gen. over $\mathrm{gr}\,\mathcal{A} \xrightarrow{\sim} A$.

Such a filtration exists but is non-unique.

$I := \mathrm{Ann}_A(\mathrm{gr}\, \mathcal{M})$ (depends on filtration, but $\sqrt{I}$ doesn't), $Y := \mathrm{Spec}(A/I)$
- coisotropic subscheme (Gabber's thm); $\{I, I\} \subset I$.

Assume:

(i) Y is generically reduced; $Y^\circ := Y^{reg} \cap X^{reg}$ is dense in Y & is Lagrangian.

Examples: 1) $\underline{X}$ is singular symplectic, $X := \underline{X} \times \underline{X}^{opp}$ (opposite bracket), $Y := \underline{X}_{diag}$, $Y^\circ = Y^{reg}$. Quant'ns of X: $\underline{\mathcal{A}}_{\lambda_1} \otimes \underline{\mathcal{A}}_{\lambda_2}^{opp}$; $\mathcal{M}$'s satisfying (i) are called Harish-Chandra $\mathcal{A}_{\lambda_1}$-$\mathcal{A}_{\lambda_2}$-bimodules.

Example of $\mathcal{M}$: regular bimodule $\underline{\mathcal{A}}_\lambda$ w. default filtration ($\lambda_i = \lambda$).

2) $X = \mathrm{Spec}\, \mathbb{C}[\mathbb{O}] \xrightarrow{\mu} \mathfrak{g}^*$ $\mathfrak{k} \subset \mathfrak{g}$ symmetric subalgebra, $Y := \mu^{-1}((\mathfrak{g}/\mathfrak{k})^*)$, $Y^\circ \supset \mathbb{O} \cap (\mathfrak{g}/\mathfrak{k})^*$, fin. union of K-orbits.
$\mathcal{M}$ satisfies (i) $\Rightarrow$ $\mathfrak{k} \curvearrowright \mathcal{M}$ is loc. fin ($\mathcal{M}$ is HC $(\mathcal{A}_\lambda, \mathfrak{k})$-module).
If $U(\mathfrak{g}) \twoheadrightarrow \mathcal{A}_\lambda$ (true e.g. for $\lambda = 0$) then $\forall$ HC $(\mathfrak{g}, \mathfrak{k})$-module killed by ker has good filtration w. (i).

$\operatorname{gr}\mathcal{M}$ comes w. additional structure (compare to $\{\cdot,\cdot\}$ on $\operatorname{gr}\mathcal{A}_\lambda$): $a \in \mathcal{A}_{\leq i}$ w. $a + \mathcal{A}_{\leq i-1} \in I$, $m \in \mathcal{M}_{\leq j} \Rightarrow am \in \mathcal{M}_{\leq i+j-1} \rightsquigarrow$ Lie algebra action of I on $\operatorname{gr}\mathcal{M}$ of deg -1, factors through $I/I^2 (\in \operatorname{Coh}(Y))$, $I/I^2|_{Y^0} \simeq T_{Y^0}$

& $(\operatorname{gr}\mathcal{M})|_{Y^0}$ is a twisted local system on Y^0 whose twist is recovered from the quantization parameter (e.g. for canonical quantization it's half-canonical. In Example 1, it's 0 if $\lambda_1 = \lambda_2$. In known situations (e.g. Examples 1 & 2), tw. local system structure on $\operatorname{gr}\mathcal{M}|_{Y^0}$ is uniquely recovered from $\mathcal{M}$.

Assume further:

ii) $\mathcal{M}$ is irreducible

Question: Can one recover $\mathcal{M}$ from $(Y^0, \mathcal{L} := (\operatorname{gr}\mathcal{M})|_{Y^0})$?

In Example 2, this will give "Orbit method."

To answer this we should impose

iii) $\operatorname{codim}_Y Y \setminus Y^0 \geq 2$

In Ex 1, iii) is the case always. In Ex 2, this is the case if $\operatorname{codim}_{\overline{\mathbb{O}}} \overline{\mathbb{O}} \setminus \mathbb{O} \geq 4$ ($\mathbb{O}' \cap (\mathfrak{g}/\mathfrak{k})^* \subset \mathbb{O}'$ is lagrangian $\forall \mathbb{O}'$).

Hopes (Thms in Ex's 1) & 2)):

- ($\mathcal{M}$ is irreducible $\Rightarrow$) $\mathcal{L}$ is irreducible
- $\mathcal{M}_1 \not\simeq \mathcal{M}_2$ (irreducible) $\Rightarrow \mathcal{L}_1 \not\simeq \mathcal{L}_2$

Existence (for irreducible twisted local system $\mathcal{L}$, is there irreducible $\mathcal{M}$ s.t. $(\operatorname{gr}\mathcal{M})|_{Y_0} \simeq \mathcal{L}$; say $\mathcal{M}$ quantizes $\mathcal{L}$)

Thm 1 (I.L. 2018):

Let $X = \underline{X} \times \underline{X}^{opp}$, $Y = \underline{X}_{diag}$, $\mathcal{A} = \underline{\mathcal{A}}_\lambda \otimes \underline{\mathcal{A}}_\lambda^{opp}$. Then $\exists!$ $\Gamma_\lambda \triangleleft \pi_1^{alg}(\underline{X}^{reg})$ (finite group) s.t. for local system $\mathcal{L}$ TFAE:

- $\exists$ $\mathcal{A}_\lambda$-bimodule $\mathcal{M}$ quantizing $\mathcal{L}$.
- The monodromy of $\mathcal{L}$ is trivial on Γ_λ.

In almost all cases one can compute Γ_λ from λ.

Example: $\underline{X} = \mathcal{N}$, $\Gamma_0 = \pi_1(\mathcal{N}^{reg}) = Z(G^{sc})$.

Thm 2 (I.L.-S. Yu, in prep.): $K :=$ symmetric subgroup in G, $\mathbb{O} \subset \mathfrak{g}^*$ w. $\operatorname{codim}_{\overline{\mathbb{O}}} \overline{\mathbb{O}} \setminus \mathbb{O} \geqslant 4$, $X = \operatorname{Spec} \mathbb{C}[\mathbb{O}]$, $Y = \mu^{-1}((\mathfrak{g}/\mathfrak{k})^*)$
Every suitably twisted local system on Y° is quantized by a HC $(\mathcal{A}_\lambda, K)$-module.

Remark:

The idea of the proof is to test whether a local system is quantized by restricting it to slices in X (of dim 4) to codim 2 strata in Y.

We can describe all such slices in settings of both Thms. Essentially, once $\mathcal{L}|_{slice}$ is quantized (by a module over a quantization of the slice), then so is $\mathcal{L}$.