Automorphic Representations and "Golden" Quantum Logic Gates

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Note on technical details

- Anything in gray is a technical detail not relevant to this particular topic
- Anything in orange we will only explain intuitively and imprecisely due to time constraints.

Outline

- Quantum Computing Motivation
- Results/Summary of Argument
- Argument step details

Draft available at: `https://www.mat.univie.ac.at/~rdalal/GoldenGatesDraft.pdf`

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Problem: Find a finite set S of "universal gates" in $PU(2^n)$ that can be multiplied to ~~realize~~ approximate any unitary matrix $\mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$.

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- In addition: approximation should be efficiently computable.

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  - C transversal for good quantum error correcting code
  - $TC_0T^{-1} \subseteq C$ for subgroup $C_0 \subseteq C$ linearly spanning $\mathrm{GL}(2^n)$
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- Previous work: only $n = 1, U(3)$ [Sar15], [PS18], [EP22],
  - Key new difficulty: failures of Ramanujan Conjecture $\implies$ automorphic bound drastically harder

Gate Set 1

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The controlled-S or CS gate

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & i \end{pmatrix} \subseteq PU(4)$$

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- 2-qubit Clifford group: best possible for QECC
- QC literature: focus of experimental implementation, growth + navigation known
- New result: covering bounds

Gate Set 2

Theorem

Let $S_{\zeta_3}(4)$ be the group of 4×4 monomial (i.e, generalized permutation) matrices with entries that are 3rd roots of unity. Define

$$C_3 := \left\langle S_{\zeta_3}(4), \quad \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ -1 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{pmatrix} \right\rangle, T := \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

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- $\mathrm{PGSp}_4(\mathbb{F}_3)$ worse than Clifford for QECC

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Rest of the talk: steps 1-4 in more detail

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  - sufficient: easy way to compute distance between vertices

Key Idea: Arithmetic Matrix Groups

Our Γ: from p-arithmetic matrix groups:

Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$ acts transitively on infinite $(p+1)$-regular tree

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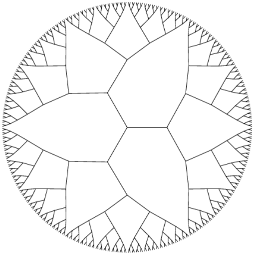

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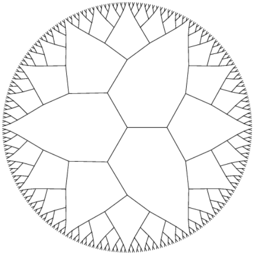

Expander Graphs Idea: $\mathrm{SL}_2\mathbb{R} \mapsto$ compact gp. allows simple action

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- $\implies$ $\mathcal{B}$ is the Cayley graph for Γ
- Gate set S: elements that take a point to its neighbors

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A compact open $K^\infty \subseteq G(\mathbb{A}^\infty)$ is *golden* if

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Key Limitation: 1 is rarely satisfied ([MSG12]: finitely many examples with rank > 4, none with rank > 8)

- future work: find all examples with rank 4

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"Super-Golden Gates"—after the next two examples, we will stick to the golden case for simplicity.

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Define unitary group G by quadratic extension and Hermitian form:

$$\mathbb{Q}(i)/\mathbb{Q}, \qquad H = \begin{pmatrix} 4 & 0 & 2+i & 1+i \\ 0 & 4 & -1+i & 2-i \\ 2-i & -1-i & 2 & 0 \\ 1-i & 2+i & 0 & 2 \end{pmatrix},$$

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$\implies$ whenever $f \in \pi^{K^\infty}$, $\mathbf{1}_{K^\infty S^{(\ell)} K^\infty} \star f = \mathbf{1}_{S^{(\ell)}} \star f$

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Interpretation: most of $\mathbf{1}_{\widetilde{B}_\delta}$ avoids violations of Ramanujan

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Upshot: rewrite bound in terms of Arthur-SL_2 instead of $\sigma(\pi, p)$.

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- $\mathbf{1}_{\widetilde{B}_\delta}$: slight modification of $\mathbf{1}_{B_\delta}$ for computational simplicity

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Final Step: plug in formulas for $d_\square(\lambda_\infty), a(\lambda_\infty,\delta)$ and sum!

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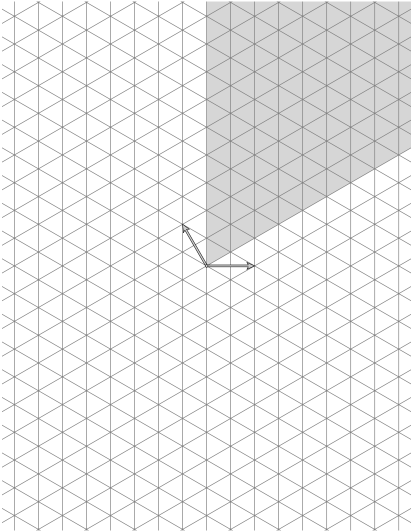
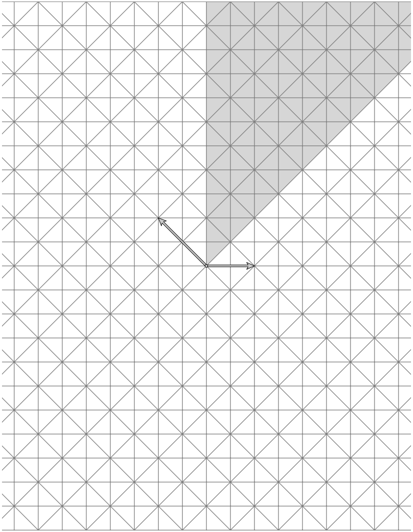
  - $\mathrm{GL}_2/\mathbb{Q}_p$: apartments are infinite two-sided paths, each $\cong \mathbb{R}$.
- If K is a maximal compact special subgroup, $G(\mathbb{Q}_p)/K$ embeds as a subset of the vertices of $\mathcal{B}$.
  - Consistent with $G(\mathbb{Q}_p)$-action
  - K is the stabilizer of fixed vertex v_0.
  - $\mathrm{GL}_2/\mathbb{Q}_p$: $G(\mathbb{Q}_p)/K$ is the vertices of the tree

Example Apartments





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- Good enough for rank-$4, 8$ after combinatorial casework on computer

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